EML 6934 – Optimal Control
Course Project

Spring 2022

1 Overview of Course Project

The project for the course is structured as follows. You are required to choose two problems from the two lists shown below. One of the problems must be from Section 2 while the second problem must be from Section 3. For the problem chosen from Section 2 you must compute either the analytic optimal solution or must compute a numerical solution using an indirect method (that is, by deriving the first-order optimality conditions and solving the Hamiltonian boundary-value problem). You must then solve the problem using one of the direct methods studied in the course. A comparison between the analytic/indirect solution and the direct solution is then required. In your analysis you must explain the key issues encountered when solving the problem using the indirect and direct approaches.

The problems from Section 3 do not have analytic solutions and cannot be easily solved using a simple indirect method (for example, indirect shooting). Consequently, it is necessary to employ a more sophisticated approach to solve these problems. Your task is to solve your chosen probem using *either* one of the indirect methods or one of the direct methods studied in the course. Regardless of the approach you choose, you must provide the following analysis of your numerical solutions. First, you must assess the proximity of your numerical solution to the "true" optimal solution (knowing full well that you do not have the "true" optimal solution to your chosen problem). In other words, how do you know you have obtained a good approximation to the true optimal? Next, you must provide a study of both the computational efficiency and robustness of your chosen method in determining the numerical approximation. In your analysis, explain how good an initial guess you need to provide in order to solve the problem and how efficiently you are able to solve the problem. In addition, analyze the limitations of the method you have chosen? Given your analysis, explain if the method you chose is the best one for this type of problem? If not, which method would be more preferable?

As as part of your analysis you can check the numerical solutions you obtain against the solutions obtained using the MATLAB optimal control software $\mathbb{GPOPS}-\mathbb{II}$. A license of $\mathbb{GPOPS}-\mathbb{II}$ is available at no charge for use at the University of Florida and can be obtained by registering on the $\mathbb{GPOPS}-\mathbb{II}$ **website** by clicking **here**. $\mathbb{GPOPS}-\mathbb{II}$ implements a variable-order Legendre-Gauss-Radau quadrature method and the details of the method can be found in the journal article that is soon to appear in the *ACM Transactions on Mathematical Software*. The final version of the journal article can be found by clicking **here**.

Finally, you must provide a comprehensive report detailing all of the results you obtained and what you learned about the numerical methods you employed. You must also provide all code used to generate your results. Please note, you must implement all numerical computations yourself and cannot use any canned software to solve your problems. The course project is due on the last day of Spring 2014 classes.

2 Elementary Optimal Control Problems

Every problem found in this list has an analytic solution. You must choose *one* problem from this Section for your project. For any problem you choose in this Section you are required to derive the optimal solution using the optimal control theory learned in the course. Next, you are required to solve your chosen problem using one of the indirect methods and one direct method studied in the course. You must then compare the quality of the numerical solutions obtained against the optimal solution. You must then analyze the quality of your numerical approximations and assess the key computational issues you encountered when trying to solve this problem using your chosen indirect and direct approach.

2.1 Hyper-Sensitive Problem

Consider the following optimal control problem taken from Ref. **?**. Minimize the cost functional

$$J = \frac{1}{2}\int_0^{t_f} (x^2 + u^2)dt \tag{1}$$

subject to the dynamic constraint

$$\dot{x} = -x + u, \tag{2}$$

the boundary conditions

$$\begin{aligned} x(0) &= 1, \\ x(t_f) &= 1.5, \end{aligned} \tag{3}$$

and t_f fixed. Solve this optimal control problem for the following values of t_f: 10, 20, 50, 100, 500, and 1000. Do you notice anything interesting in your ability to compute the both the analytic solution on your computer and the numerical solution as t_f increases? Describe your observation.

2.2 Linear Tangent Steering Problem

Consider the following optimal control problem. Minimize the cost functional

$$J = t_f \tag{4}$$

subject to the dynamic constraints

$$\begin{aligned} \dot{x}_1 &= x_3, \\ \dot{x}_2 &= x_4, \\ \dot{x}_3 &= a\cos u, \\ \dot{x}_4 &= a\sin u, \end{aligned} \tag{5}$$

and the boundary conditions

$$\begin{aligned} x_1(0) &= 0, \\ x_2(0) &= 0, \\ x_3(0) &= 0, \\ x_4(0) &= 0, \\ x_2(t_f) &= 5, \\ x_3(t_f) &= 45, \\ x_4(t_f) &= 0. \end{aligned} \tag{6}$$

2.3 Ground Mobile Robot Problem

The following optimal control problem is originally found in Ref. **?** and corresponds to the minimum time transfer of a ground mobile between a given initial and terminal state. Minimize the cost functional

$$J = t_f \tag{7}$$

subject to the dynamic constraints

$$\begin{aligned} \dot{x} &= \cos(\theta), \\ \dot{y} &= \sin(\theta), \\ \dot{\theta} &= u, \end{aligned} \tag{8}$$

and the boundary conditions

$$\begin{aligned} x(0) &= 0, \\ y(0) &= 0, \\ \theta(0) &= -\pi, \\ x(t_f) &= 0, \\ y(t_f) &= 0, \\ \theta(t_f) &= \pi. \end{aligned} \tag{9}$$

2.4 Moon Lander Problem

This optimal control problem was originally posed by Meditch.[?] The objective is to attain a soft landing on moon during vertical descent from an initial altitude and velocity above the lunar surface. The problem is stated as follows. Minimize the cost functional

$$J = \int_0^{t_f} u dt \tag{10}$$

subject to the dynamic constraints

$$\begin{aligned} \dot{h} &= v, \\ \dot{v} &= -g + u, \end{aligned} \tag{11}$$

the boundary conditions

$$\begin{aligned} h(0) &= 10, \\ v(0) &= -2, \\ h(t_f) &= 0, \\ v(t_f) &= 0, \end{aligned} \tag{12}$$

and the control inequality constraint

$$u_{\min} \le u \le u_{\max}, \tag{13}$$

where $u_{\min} = 0$, $u_{\max} = 3$, $g = 1.5$, and t_f is free.

2.5 Bryson-Denham Problem

Consider the following optimal control minimum-energy optimal control problem with an inequality state constraint taken from Ref. **?**. Minimize the cost functional

$$J = \frac{1}{2}\int_0^1 a^2 dt \tag{14}$$

subject to the dynamic constraints

$$\begin{aligned} \dot{x} &= v, & (15)\\ \dot{v} &= a, & (16) \end{aligned}$$

the boundary conditions

$$\begin{aligned} x(0) &= x(1) = 0, & (17)\\ v(0) &= -v(1) = 1, & (18) \end{aligned}$$

and the constraint $x(t) \le \ell$.

3 Advanced Optimal Control Problems

No problem in this Section has an analytic solution. As a result, every problem must be solved numerically. As with the elementary problems found in Section 2, the problem you choose in this Section must be solved using either one of the indirect methods or one of the direct methods studied in the course. You must then provide the following analysis of your numerical approximations. First, provide an assessment of the proximity of your numerical solutions to the optimal solution (which, as stated, is not known for these problems). How do you know you have obtained a reasonable approximation? Next, what is the computational efficiency of the numerical methods you used to solve your problem? What are the limitations of the methods you have chosen on your problem. Given your analysis, what numerical method would you seek in order to overcome the deficiencies you found with the methods you chose?

3.1 Crossrange Maximization During Entry of a Reusable Launch Vehicle

The reusable launch vehicle entry problem is taken from Ref. **?**. The objective of the problem is to maximize the crossrange subtended by the vehicle during entry, where the entry starts at the edge of the sensable atmosphere and terminates at the start of the *terminal area energy management* (TAEM) phase. The problem is stated as follows. Maximize the objective functional

$$J = \phi(t_f) \tag{19}$$

subject to the dynamic constraints

$$\begin{aligned}
\dot{r} &= v \sin\gamma, \\
\dot{\theta} &= \frac{v\cos\gamma\sin\psi}{r\cos\phi}, \\
\dot{\phi} &= \frac{v\cos\gamma\cos\psi}{r}, \\
\dot{v} &= -\frac{D}{m} - g\sin\gamma, \\
\dot{\gamma} &= \frac{L\cos\sigma}{mv} - \left(\frac{g}{v} - \frac{v}{r}\right)\cos\gamma, \\
\dot{\psi} &= \frac{L\sin\sigma}{mv\cos\gamma} + \frac{v\cos\gamma\sin\psi\tan\phi}{r},
\end{aligned} \tag{20}$$

and the boundary conditions

$$\begin{array}{rclcrcl}
h(0) &=& 792.48\text{ km} &,& h(t_f) &=& 24.384\text{ km}, \\
\theta(0) &=& 0\text{ deg} &,& \theta(t_f) &=& \text{Free}, \\
\phi(0) &=& 0\text{ deg} &,& \phi(t_f) &=& \text{Free}, \\
v(0) &=& 7802.88\text{ m/s} &,& v(t_f) &=& 762.0\text{ m/s}, \\
\gamma(0) &=& -1\text{ deg} &,& \gamma(t_f) &=& -5\text{ deg}, \\
\psi(0) &=& 90\text{ deg} &,& \psi(t_f) &=& \text{Free},
\end{array} \tag{21}$$

where $r = h + R_e$ is the geocentric radius, h is the altitude, R_e is the polar radius of the Earth, θ is the longitude, ϕ is the latitude, v is the speed, γ is the flight path angle, and ψ is the azimuth angle. Furthermore, the aerodynamic and gravitational forces are computed as

$$\begin{aligned}
D &= \rho v^2 S C_D/2, \\
L &= \rho v^2 S C_L/2, \\
g &= \mu/r^2,
\end{aligned} \tag{22}$$

where $\rho = \rho_0 \exp(-h/H)$ is the atmospheric density, ρ_0 is the density at sea level, H is the density scale height, S is the vehicle reference area, C_D is the coefficient of drag, C_L is the coefficient of lift, and μ is the gravitational parameter. The coefficient of lift and drag are computed, respectively, as

$$C_D = C_{D0} + C_{D1}\alpha + C_{D2}\alpha^2, \tag{23}$$

$$C_L = C_{L0} + C_{L1}\alpha, \tag{24}$$

where α is the angle of attack. Table 1 provides the constants used in this problem.

3.2 Maximization of Mass-to-Orbit for a Multiple-Stage Launch Vehicle

The problem considered in this section is the ascent of a multiple-stage launch vehicle. The objective is to maneuver the launch vehicle from the ground to the target orbit while maximizing the remaining fuel in the upper stage. It is noted that this example is is found verbatim in Refs. **?**, **?**, **?**. The problem is modeled using four phases where the objective is to maximize the mass at the end of the fourth phase, that is maximize

$$J = m(t_f^{(4)}) \tag{25}$$

Table 1: Constants for Reusable Launch Vehicle Entry Problem.

Quantity	Value
R_e	6.37120392×10^6 m
S	249.9091776 m^2
C_{D0}	0.0785
C_{D1}	-0.3529 rad^{-1}
C_{D2}	2.04 rad^{-2}
C_{L0}	-0.207
C_{L1}	1.6756 rad^{-1}
H	7254.24 m
ρ_0	1.225571 kg·m^{-3}
μ	3.986032×10^{14} m^3·s^{-2}
m	92079.253 kg

subject to the dynamic constraints

$$\begin{array}{rcl} \dot{\mathbf{r}}^{(p)} & = & \mathbf{v}^{(p)}, \\ \dot{\mathbf{v}}^{(p)} & = & -\dfrac{\mu}{\|\mathbf{r}^{(p)}\|^3}\mathbf{r}^{(p)} + \dfrac{T^{(p)}}{m^{(p)}}\mathbf{u}^{(p)} + \dfrac{\mathbf{D}^{(p)}}{m^{(p)}}, \qquad (p = 1, \ldots, 4), \\ \dot{m}^{(p)} & = & -\dfrac{T^{(p)}}{g_0 I_{sp}}, \end{array} \tag{26}$$

the initial conditions

$$\begin{array}{rcl} \mathbf{r}(t_0) & = & \mathbf{r}_0 = (5605.2, 0, 3043.4) \times 10^3 \text{ m}, \\ \mathbf{v}(t_0) & = & \mathbf{v}_0 = (0, 0.4076, 0) \times 10^3 \text{ m/s}, \\ m(t_0) & = & m_0 = 301454 \text{ kg}. \end{array} \tag{27}$$

the interior point constraints

$$\begin{array}{rcl} \mathbf{r}^{(p)}(t_f^{(p)}) - \mathbf{r}^{(p+1)}(t_0^{(p+1)}) & = & \mathbf{0}, \\ \mathbf{v}^{(p)}(t_f^{(p)}) - \mathbf{v}^{(p+1)}(t_0^{(p+1)}) & = & \mathbf{0}, \qquad (p = 1, \ldots, 3) \\ m^{(p)}(t_f^{(p)}) - m_{\text{dry}}^{(p)} - m^{(p+1)}(t_0^{(p+1)}) & = & 0, \end{array} \tag{28}$$

the terminal constraints (corresponding to a geosynchronous transfer orbit),

$$\begin{array}{rclcrcl} a(t_f^{(4)}) & = & a_f = 24361.14 \text{ km} & , & e(t_f^{(4)}) & = & e_f = 0.7308, \\ i(t_f^{(4)}) & = & i_f = 28.5 \deg & , & \theta(t_f^{(4)}) & = & \theta_f = 269.8 \deg, \\ & & \phi(t_f^{(4)}) = \phi_f = 130.5 \deg, & & & & \end{array} \tag{29}$$

and the path constraints

$$\begin{array}{rcl} \|\mathbf{r}^{(p)}\|_2^2 & \geq & R_e, \\ \|\mathbf{u}^{(p)}\|_2^2 & = & 1, \end{array} \qquad (p = 1, \ldots, 4). \tag{30}$$

In each phase $\mathbf{r}(t) = (x(t), y(t), z(t))$ is the position relative to the center of the Earth expressed in ECI coordinates, $\mathbf{v} = (v_x(t), v_y(t), v_z(t))$ is the inertial velocity expressed in ECI coordinates, μ

is the gravitational parameter, T is the vacuum thrust, m is the mass, g_0 is the acceleration due to gravity at sea level, I_{sp} is the specific impulse of the engine, $\mathbf{u} = (u_x, u_y, u_z)$ is the thrust direction expressed in ECI coordinates, and $\mathbf{D} = (D_x, D_y, D_z)$ is the drag force expressed ECI coordinates. The drag force is defined as

$$\mathbf{D} = -\tfrac{1}{2} C_D S \rho \|\mathbf{v}_{\text{rel}}\| \mathbf{v}_{\text{rel}} \tag{31}$$

where C_D is the drag coefficient, S is the vehicle reference area, $\rho = \rho_0 \exp(-h/H)$ is the atmospheric density, ρ_0 is the sea level density, $h = r - R_e$ is the altitude, $r = \|\mathbf{r}\|_2 = \sqrt{x^2 + y^2 + z^2}$ is the geocentric radius, R_e is the equatorial radius of the Earth, H is the density scale height, and $\mathbf{v}_{\text{rel}} = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}$ is the velocity as viewed by an observer fixed to the Earth expressed in ECI coordinates, and $\boldsymbol{\omega} = (0, 0, \Omega)$ is the angular velocity of the Earth as viewed by an observer in the inertial reference frame expressed in ECI coordinates. Furthermore, m_{dry} is the dry mass of phases 1, 2, and 3 and is defined $m_{\text{dry}} = m_{\text{tot}} - m_{\text{prop}}$, where m_{tot} and m_{prop} are, respectively, the total mass and dry mass of phases 1, 2, and 3. Finally, the quantities a, e, i, θ, and ϕ are, respectively, the semi-major axis, eccentricity, inclination, longitude of ascending node, and argument of periapsis, respectively. The vehicle data for this problem and the the numerical values for the physical constants can be found in Tables 2 and 3, respectively.

Table 2: Vehicle Properties for Multiple-Stage Launch Vehicle Ascent Problem.

Quantity	Solid Boosters	Stage 1	Stage 2
m_{tot} (kg)	19290	104380	19300
m_{prop} (kg)	17010	95550	16820
T (N)	628500	1083100	110094
I_{sp} (s)	283.3	301.7	467.2
Number of Engines	9	1	1
Burn Time (s)	75.2	261	700

Table 3: Constants Used in the Launch Vehicle Ascent Optimal Control Problem.

Constant	Value
Payload Mass	4164 kg
S	4π m^2
C_D	0.5
ρ_0	1.225 kg/m^3
H	7200 m
t_1	75.2 s
t_2	150.4 s
t_3	261 s
R_e	6378145 m
Ω	$7.29211585 \times 10^{-5}$ rad/s
μ	3.986012×10^{14} m^3/s^2
g_0	9.80665 m/s^2

3.3 Minimum Time-to-Climb of a Supersonic Aircraft

An extremely famous optimal control problem is the *minimum time-to-climb of a supersonic aircraft*. This problem was originally solved for the F-4 aircraft in Ref. **?**. In this Section we describe a variation of this famous problem using a model developed in Ref. **?** for a more modern supersonic aircraft. The objective of the problem is to minimize the cost functional

$$J = t_f \tag{32}$$

subject to the dynamic constraints

$$\begin{array}{rcl} \dot{h} & = & v \sin\gamma, \\ \dot{v} & = & \frac{T-D}{m} - g\sin\gamma, \\ \dot{\gamma} & = & \frac{g}{v}(u - \cos\gamma), \end{array} \tag{33}$$

and the boundary conditions

$$\begin{array}{rclcrcl} h(0) & = & 0 & , & h(t_f) & = & 20000 \text{ m} \\ v(0) & = & 129 \text{ m}\cdot\text{s}^{-1} & , & v(t_f) & = & 295 \text{ m}\cdot\text{s}^{-1}, \\ \gamma(0) & = & 0 & , & \gamma(t_f) & = & 0, \end{array} \tag{34}$$

where h is the altitude, v is the speed, γ is the flight path angle, T is the thrust force, D is the drag force, g is the acceleration due to gravity, and u is the load factor (and is the control). The thrust and drag are computed as

$$T = T(h, M) = \frac{9.80665}{2.2}\sum_{i=0}^{5} e_i(M)\bar{h}^i, \tag{35}$$

$$D = q\left[C_{D0}(M) + K(M)\left(\frac{mgu}{q}\right)^2\right], \tag{36}$$

$$\tag{37}$$

where $\bar{h} = h/1000$, $q = \rho v^2/2$ is the dynamic pressure, $M = v/a$ is the Mach number, $a = a_0\sqrt{\theta}$ is the speed of sound, $\rho = \rho_0 \exp(y)$, and

$$y = y_0 + y_1\bar{h} + r, \tag{38}$$

$$r = r_0 \exp(-z), \tag{39}$$

$$z = \sum_{j=1}^{4} z_j \bar{h}^j, \tag{40}$$

$$\theta = \sum_{j=0}^{3} \theta_j \bar{h}^j, \tag{41}$$

$$C_{D0} = \frac{\sum_{i=0}^{4} a_i M^i}{\sum_{j=0}^{4} b_i M^i}, \tag{42}$$

$$K = \frac{\sum_{i=0}^{4} c_i M^i}{\sum_{i=0}^{5} d_i M^i}, \tag{43}$$

$$e_i = \sum_{j=0}^{5} f_{ji} M^j, \quad (i = 0, 1, \ldots, 5). \tag{44}$$

	a_j	b_j	c_j	d_j
$j=0$	$+2.61059846050 \times 10^{-2}$	$+1.37368651246 \times 10^{0}$	$+1.23001735612 \times 10^{0}$	$+1.42392902737 \times 10^{1}$
$j=1$	$-8.57043966269 \times 10^{-2}$	$-4.57116286752 \times 10^{0}$	$-2.97244144190 \times 10^{0}$	$-3.24759126471 \times 10^{1}$
$j=2$	$+1.07863115049 \times 10^{-1}$	$+5.72789877344 \times 10^{0}$	$+2.78009092756 \times 10^{0}$	$+2.96838743792 \times 10^{1}$
$j=3$	$-6.44772018636 \times 10^{-2}$	$-3.25219000620 \times 10^{0}$	$-1.16227834301 \times 10^{0}$	$-1.33316812491 \times 10^{1}$
$j=4$	$+1.64933626507 \times 10^{-2}$	$+7.29821847445 \times 10^{-1}$	$+1.81868987624 \times 10^{-1}$	$+2.87165882405 \times 10^{0}$
$j=5$	$--$	$--$	$--$	$-2.27239723756 \times 10^{-1}$

Table 4: Coefficients y_i, z_i, and θ_i.

	y_i	z_i	θ_i
$i=0$	-1.02280550	$--$	$+292.1000$
$i=1$	-0.12122693	-0.03486432410000	-8.877430
$i=2$	$--$	$+0.03509918650000$	$+0.193315$
$i=3$	$--$	-0.00008330005350	$+0.003720$
$i=4$	$--$	$+0.00000115219733$	$--$

Table 5: Coefficients r_0 and ρ_0.

Quantity	Value
r_0	1.0228055
ρ_0 (kg·m^{-3})	1.2250000

Table 6: Coefficients f_{ji}, $(j, i = 0, \ldots, 5)$.

$j \backslash i$	f_{j0}	f_{j1}	f_{j2}	f_{j3}	f_{j4}	f_{j5}
$j=0$	$+119699.95703$	-14644.656421	-455.34597613	$+495.44694509$	-46.253181596	$+1.2000480258$
$j=1$	-352173.18620	$+51808.811078$	$+2314.3969006$	-2248.2310455	$+208.94683419$	-5.3807416658
$j=2$	$+604521.59152$	-95597.112936	-3886.0323817	$+3977.1922607$	-368.35984294	$+9.4529288471$
$j=3$	-430429.85701	$+83271.826575$	$+1235.7128390$	-3073.4191752	$+293.88870979$	-7.6204728620
$j=4$	$+136569.37908$	-32867.923740	$+555.72727442$	$+1063.5494768$	-107.84916936	$+2.8552696781$
$j=5$	-16647.992124	$+4910.2536402$	-235.91380327	-136.26703723	$+14.880019422$	-0.40379767869

3.4 Two-Strain Tuberculosis Problem

Ref. **?** describe a model for two-strain tuberculosis treatment as follows:

> In the absence of an effective vaccine, current control programs for TB have focused on chemotherapy. The antibiotic treatment for an active TB (with drug-sensitive strain) patient requires a much longer period of time and a higher cost than that for those who are infected with sensitive TB but have not developed the disease. Lack of compliance with drug treatments not only may lead to a relapse but to the development of antibiotic resistant TB — one of the most serious public health problems facing society today. A report released by the World Health Organization warns that if countries do not act quickly to strengthen their control of TB, the multi-drug resistant strains that have cost New York City and Russia hundreds of lives and more than $1 billion each will continue to emerge in other parts of the world. The reduction in cases of drug sensitive TB can be achieved either by "case holding," which refers to activities and techniques used to ensure regularity of drug intake for a duration adequate to achieve a cure, or by "case finding," which refers to the identification (through screening, for example) of individuals latently infected with sensitive TB who are at high risk of developing the disease and who may benefit from preventive intervention. These preventive treatments will reduce the incidence (new cases per unit of time) of drug sensitive TB and hence indirectly reduce the incidence of drug resistant TB.

The dynamic model divides the host population into distinct epidemiological classes, in which the population of each class is treated as a state variable. Thus, the total population satisfies the relation

$$N = S + L_1 + I_1 + L_2 + I_2 + T \tag{45}$$

where the state is $(S, T, L_1, I_1, L_2, I_2)$ such that S are those that are susceptible to TB, T are those who are treated effectively, L_1 are those who are infected with latent typical non-infectious TB, L_2 are those who are infected with resistant but non-infectious TB, I_1 are those are infected with typical infectious TB, and I_2 are those whos are infected resistant TB. The dynamics of the system are given as

$$\begin{aligned}
\dot{S} &= \mu N - \beta_1 S \tfrac{I_1}{N} - \beta^* \tfrac{I_2}{N} - \mu S \\
\dot{T} &= u_1 r_1 L_1 - \mu T + (1 - (1 - u_2)(p + q)) r_2 I_1 - \beta_2 T \tfrac{I_1}{N} - \beta^* T \tfrac{I_2}{N} \\
\dot{L}_1 &= \beta_1 S \tfrac{I_1}{N} - (\mu + k_1) L_1 - u_1 r_1 L_1 + (1 - u_2) p r_2 I_1 + \beta_2 T \tfrac{I_1}{N} - \beta^* L_1 \tfrac{I_2}{N} \\
\dot{L}_2 &= (1 - u_2) q r_2 I_1 - (\mu + k_2) L_2 + \beta^* (S + L_1 + T) \tfrac{I_2}{N} \\
\dot{I}_1 &= k_1 L_1 - (\mu + d_1) I_1 - r_2 I_1 \\
\dot{I}_2 &= k_2 L_2 - (\mu + d_2) I_2
\end{aligned} \tag{46}$$

The objective is then to minimize the cost functional

$$J = \int_0^{t_f} \left[L_2 + I_2 + \frac{1}{2} B_1 u_1^2 + \frac{1}{2} B_2 u_2^2 \right] dt \tag{47}$$

subject to the dynamic constraints of Eq. (46), the equality path constraint of Eq. (45), and the initial conditions

$$\begin{array}{rcl} L(0) & = & 76N/120, \\ T(0) & = & N/120, \\ L_1(0) & = & 36N/120, \\ L_2(0) & = & 2N/120, \\ I_1(0) & = & 4N/120, \\ I_2(0) & = & N/120, \end{array} \tag{48}$$

where $t_f = 5$ and the constants used for this problem are found in Table 7. Note for this problem that the components of the state take on real values even though each state component represents a quantity that in reality must take on an integer value.

Table 7: Constants Used in Tuberculosis Optimal Control Problem.

Quantity	Value
β_1	13
β_2	13
μ	0.0143
d_1	0
d_2	0
k_1	0.5
k_2	1
r_1	2
r_2	1
p	0.4
q	0.1
N	30000
β^*	0.029
B_1	50
B_2	500

3.5 Robot Arm Problem

The robot arm optimal control problem was originally posed and solved by Elizabeth Dolan and Jorge J. More of Argonne National Laboratory. The objective of the problem is to minimize the time taken to reorient the arm from an initial orientation to a final orientation. The optimal control problem is stated as follows. Minimize the cost functional

$$J = t_f \tag{49}$$

subject to the dynamic constraints

$$\begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & u_1/L, \\ \dot{x}_3 & = & x_4, \\ \dot{x}_4 & = & u_2/I_\theta, \\ \dot{x}_5 & = & x_6, \\ \dot{x}_6 & = & u_3/I_\phi, \end{array} \tag{50}$$

the control inequality constraints

$$|u_i| \leq 1, \quad (i = 1, 2, 3), \tag{51}$$

and the boundary conditions

$$\begin{array}{lllllll} t_0 & = & 0 & , & t_f & = & \text{Free}, \\ x_1(t_0) & = & 4.5 & , & x_1(t_f) & = & 4.5, \\ x_2(t_0) & = & 0 & , & x_2(t_f) & = & 0, \\ x_3(t_0) & = & 0 & , & x_3(t_f) & = & 2\pi/3, \\ x_4(t_0) & = & 0 & , & x_4(t_f) & = & 0, \\ x_5(t_0) & = & \pi/4 & , & x_5(t_f) & = & \pi/4, \\ x_6(t_0) & = & 0 & , & x_6(t_f) & = & 0, \end{array} \tag{52}$$

where

$$\begin{array}{lll} I_\phi & = & \frac{1}{3}\left[(L - x_1)^3 + x_1^3\right], \\ I_\theta & = & I_\phi \sin^2(x_5), \end{array} \tag{53}$$

and $L = 5$.