EGM 3401

Theory Assignment #3

Spring 2021

Question 1

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors $\mathbb{E}^3$ and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a right-handed orthonormal basis for $\mathbb{E}^3$. Furthermore, let $\mathbf{a} \times \mathbf{b}$ be the vector product between $\mathbf{a}$ and $\mathbf{b}$. Derive an expression for the tensor $\mathbf{a}^{\times} \cdot \mathbf{b} = \mathbf{a} \times \mathbf{b}$ in terms of the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, that is derive an expression for $\mathbf{a}^{\times}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which has the form

$$\mathbf{T} = \sum_{i=1}^{3} \sum_{j=1}^{3} T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j,$$

where $\mathbf{T} = \mathbf{a}^{\times}$.

Question 2

Consider a rigid body $\mathcal{R}$. Prove the following statements:

(a) $m\bar{\mathbf{r}} - \int_{\mathcal{R}} \mathbf{r} dm = \mathbf{0}$,

(b) $m^{\mathcal{N}}\bar{\mathbf{v}} - \int_{\mathcal{R}} {}^{\mathcal{N}}\mathbf{v} dm = \mathbf{0}$,

(c) $m^{\mathcal{N}}\bar{\mathbf{a}} - \int_{\mathcal{R}} {}^{\mathcal{N}}\mathbf{a} dm = \mathbf{0}$,

where $(\bar{\mathbf{r}}, {}^{\mathcal{N}}\bar{\mathbf{v}}, {}^{\mathcal{N}}\bar{\mathbf{a}})$, are, respectively, the position, velocity, and acceleration of the center of mass of $\mathcal{R}$.

Question 3

Let $\boldsymbol{\tau}$ be a pure torque applied to a rigid body $\mathcal{R}$. Prove that the torque $\boldsymbol{\tau}$ is a free vector and can thus be transported between two points P anq Q on $\mathcal{R}$ without changing the torque.

Question 4

The moment due to a system of forces $(\mathbf{F}_1, \ldots, \mathbf{F}_n)$ and a pure torque $\boldsymbol{\tau}$ applied to a rigid body $\mathcal{R}$ relative to a point Q fixed in $\mathcal{R}$ is defined as

$$\mathbf{M}_Q = \sum_{i=1}^{N} (\mathbf{r}_i - \mathbf{r}_Q) \times \mathbf{F}_i + \boldsymbol{\tau}$$

Show that $\mathbf{M}_Q$ is related to $\mathbf{M}_P$ (where $\mathbf{M}_P$ is the moment due the system of forces $(\mathbf{F}_1, \ldots, \mathbf{F}_n)$ and the pure torque $\boldsymbol{\tau}$ relative to a point P) via

$$\mathbf{M}_P = \mathbf{M}_Q + (\mathbf{r}_Q - \mathbf{r}_P) \times \mathbf{F},$$

where $\mathbf{F}$ is the resultant force acting on the rigid body.

Question 5

The angular momentum of a rigid body $\mathcal{R}$ relative to an arbitrary point Q in an inertial reference frame $\mathcal{N}$ is defined as

$${}^{\mathcal{N}}\mathbf{H}_Q = \int_{\mathcal{R}} (\mathbf{r} - \mathbf{r}_Q) \times ({}^{\mathcal{N}}\mathbf{v} - {}^{\mathcal{N}}\mathbf{v}_Q) dm$$

Suppose now that the point Q is equal to a point B where B is fixed in $\mathcal{R}$. Prove that the angular momentum relative to point B is given as

$${}^{\mathcal{N}}\mathbf{H}_B = \mathbf{I}_B^{\mathcal{R}} \cdot {}^{\mathcal{N}}\boldsymbol{\omega}^{\mathcal{R}},$$

where $\mathbf{I}_B^{\mathcal{R}}$ is the moment of inertia tensor of the rigid body $\mathcal{R}$ relative to point B and ${}^{\mathcal{N}}\boldsymbol{\omega}^{\mathcal{R}}$ is the angular velocity of $\mathcal{R}$ as viewed by an observer in the inertial reference frame $\mathcal{N}$.

Question 6

Let $\mathcal{R}$ be a rigid body, and let ${}^{\mathcal{N}}\mathbf{H}_Q$ be the angular momentum of $\mathcal{R}$ relative to an arbitrary point Q. Starting with the definition of the angular momentum of a rigid body relative to Q, that is,

$$ {}^{\mathcal{N}}\mathbf{H}_Q = \int_{\mathcal{R}} (\mathbf{r} - \mathbf{r}_Q) \times ({}^{\mathcal{N}}\mathbf{v} - {}^{\mathcal{N}}\mathbf{v}_Q) dm, $$

prove that

$$ {}^{\mathcal{N}}\mathbf{H}_Q = {}^{\mathcal{N}}\bar{\mathbf{H}} + (\mathbf{r}_Q - \bar{\mathbf{r}}) \times m({}^{\mathcal{N}}\mathbf{v}_Q - {}^{\mathcal{N}}\bar{\mathbf{v}}). $$

where ${}^{\mathcal{N}}\bar{\mathbf{H}}$ is the angular momentum of $\mathcal{R}$ relative to the center of mass of $\mathcal{R}$.

Question 7

Let $\mathcal{R}$ be a rigid body and let $\mathcal{N}$ be an inertial reference frame. Starting with the fundamental form of Euler's second law, that is,

$$ \frac{{}^{\mathcal{N}}d}{dt}\left({}^{\mathcal{N}}\mathbf{H}_O\right) = \mathbf{M}_O, $$

prove the following two results.

(a) If the reference point is the arbitary point Q, then

$$ \frac{{}^{\mathcal{N}}d}{dt}\left({}^{\mathcal{N}}\mathbf{H}_Q\right) = \mathbf{M}_Q - (\bar{\mathbf{r}} - \mathbf{r}_Q) \times m\,{}^{\mathcal{N}}\mathbf{a}_Q. $$

(b) If the reference point is the center of mass of $\mathcal{R}$, then

$$ \frac{{}^{\mathcal{N}}d}{dt}\left({}^{\mathcal{N}}\bar{\mathbf{H}}\right) = \bar{\mathbf{M}}. $$

The quantities ${}^{\mathcal{N}}\mathbf{H}_Q$ and ${}^{\mathcal{N}}\bar{\mathbf{H}}$ are, respectively, the angular momentum of the rigid body relative to an arbitrary point Q and the center of mass of $\mathcal{R}$.

Question 8

The kinetic energy of a rigid body is defined as

$$ T = \frac{1}{2}\int_{\mathcal{R}} {}^{\mathcal{N}}\mathbf{v} \cdot {}^{\mathcal{N}}\mathbf{v} dm, $$

where the integral is taken over all material points in the body. Prove that T can be written as

$$ T = \frac{1}{2}{}^{\mathcal{N}}\bar{\mathbf{v}} \cdot {}^{\mathcal{N}}\bar{\mathbf{v}} + \frac{1}{2}{}^{\mathcal{N}}\bar{\mathbf{H}} \cdot {}^{\mathcal{N}}\boldsymbol{\omega}^{\mathcal{R}}. $$

This last result is called *Koenig's decomposition* for the kinetic energy of a rigid body.

Question 9

Consider a system of n particles with mass $(m_1, \ldots, m_n)$. Suppose further that the position measured from a point O fixed in an inertial reference frame $\mathcal{N}$ can be expressed as $\mathbf{r}_i = \bar{\mathbf{r}} + \boldsymbol{\rho}_i$, where $\bar{\mathbf{r}}$ is the position of the center of mass of the system. In addition, assume that the distance between each of the particles is a constant. Using the definition of angular momentum for the center of mass of a system of particles, show that

$$^{\mathcal{N}}\bar{\mathbf{H}} = \left[\sum_{i=1}^{n} m_i \left\{ (\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i)\mathbf{U} - \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \right\} \right] \cdot {}^{\mathcal{N}}\boldsymbol{\omega}^{\mathcal{B}},$$

where $\mathcal{B}$ is the reference frame in which the particles are fixed and $\mathbf{U}$ is the identity tensor (that is, $\mathbf{U} \cdot \mathbf{a} = \mathbf{a}$).

Question 10

Let $\mathcal{R}$ be a rigid body. Show that the angular momentum of the rigid body relative to the center of mass can be written as

$$^{\mathcal{N}}\bar{\mathbf{H}} = \left[\int_{\mathcal{R}} \left\{ (\boldsymbol{\rho} \cdot \boldsymbol{\rho})\mathbf{U} - \boldsymbol{\rho} \otimes \boldsymbol{\rho} \right\} dm \right] \cdot {}^{\mathcal{N}}\boldsymbol{\omega}^{\mathcal{R}},$$

where $\mathbf{U}$ is the identity tensor.

Question 11

Let ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}$ be the angular velocity of reference frame $\mathcal{B}$ relative to reference frame $\mathcal{A}$. Show that the operation of taking the vector product of ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}$ with an arbitrary vector $\mathbf{b}$ is a tensor and show the matrix representation of this tensor in a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is fixed in $\mathcal{B}$.

Question 12

Let ${}^{\mathcal{N}}\mathbf{H}_Q$ be the angular momentum of a rigid body $\mathcal{R}$ relative to an arbitrary point Q. Prove that

$$\frac{^{\mathcal{N}}d}{dt}\left({}^{\mathcal{N}}\mathbf{H}_Q\right) = \frac{^{\mathcal{R}}d}{dt}\left({}^{\mathcal{N}}\mathbf{H}_Q\right) + {}^{\mathcal{N}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{N}}\mathbf{H}_Q.$$