ORBITAL MECHANICS: AN INTRODUCTION

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Contents

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I dedicate this book to all of the students whom I have taught and who have inspired me throughout my years as a teacher. Without all of your enthusiasm, dedication, and effort, an exposition such as this one is simply not possible.

Chapter 1

Two-Body Problem

1.1 Introduction

The starting point for astrodynamics is the study of the classical two-body problem. The two-body problem consists of a spacecraft in motion relative to a planet. Both the spacecraft and the planet are modeled as a point mass, thereby assuming that the planet exerts a central body gravitational force on the spacecraft. In this chapter the key differential equation, called the *two-body differential equation*, is derived under the assumption that the mass of the planet is significantly larger than the mass of the spacecraft. This assumption leads to an approximation that the center of the planet is an inertially fixed point, thereby approximating the planet as an inertial reference frame. It is then shown that a vector, called the *specific angular momentum*, is fixed in the inertial frame, thus leading to the fact that the solution lies in an inertially fixed plane called the *orbit plane*. It is then shown that a second vector, called the *eccentricity vector*, lies in the orbital plane and defines a direction from which the location of the spacecraft is measured. The two-body differential equation is then solved as a function of an angle, called the *true anomaly*, between the position of the spacecraft and the eccentricity vector. Several key quantities are then defined that assist in developing an understanding the motion of the spacecraft. Finally, some key properties of the solution of the two-body differential equation are derived.

1.2 Two-Body Differential Equation

Consider a system consisting of a planet of mass M and a spacecraft of mass m. The center of the planet is located at point P while the spacecraft is located at point S. While both the planet and the spacecraft have nonzero sizes, in the development presented here both objects are modeled as point masses (particles). Suppose now that both the planet and the spacecraft move relative to an inertial reference frame $\mathcal{I}$. Furthermore, let $\mathbf{r}_P$ and $\mathbf{r}_S$ be the position of the planet and the spacecraft relative to a point O, where O is fixed in $\mathcal{I}$. Then the velocity and acceleration of the planet and the spacecraft relative to the inertial reference frame $\mathcal{I}$ are given, respectively, as

$$\begin{aligned} {}^{\mathcal{I}}\mathbf{v}_P &= \frac{{}^{\mathcal{I}}d\mathbf{r}_P}{dt} \quad , \quad {}^{\mathcal{I}}\mathbf{a}_P = \frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}_P\right), \\ {}^{\mathcal{I}}\mathbf{v}_S &= \frac{{}^{\mathcal{I}}d\mathbf{r}_S}{dt} \quad , \quad {}^{\mathcal{I}}\mathbf{a}_S = \frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}_S\right). \end{aligned} \tag{1.1}$$

A schematic of the system consisting of the spacecraft and planet is shown in Fig. 1.1, where the right-handed orthonormal basis $\left\{\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z\right\}$ is fixed in the inertial reference frame $\mathcal{I}$.* Assume now that the only force acting on either the planet or the spacecraft is that due to gravitational attraction applied by the other object (that is, the only force acting on the spacecraft is that of gravitational attraction applied by the planet and, vice versa, the only force acting on the planet is that of gravitational attraction applied by the spacecraft). Under these assumptions, the objective is to determine the differential equation of motion of the spaceaft relative to the planet.

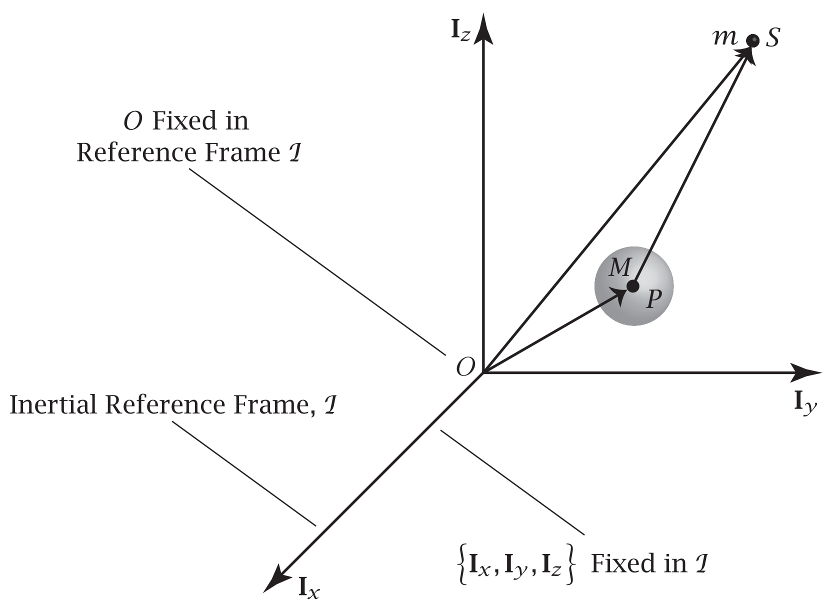


Figure 1.1 Schematic of two-body problem consisting of a planet of mass M and a spacecraft of mass m, where both the planet and the spacecraft are in motion relative to an inertial reference frame $\mathcal{I}$.

In order to determine the differential equation of motion of the spacecraft relative to the planet the following additional notation will be helpful. Let $\mathbf{r} = \mathbf{r}_S - \mathbf{r}_P$ be the position of the spacecraft relative to the planet and let $r = \|\mathbf{r}\|$ be the magnitude of $\mathbf{r}$ (that is, r is the distance between P and S and will be referred to henceforth as the *radial* distance from P to S). Then, from Newton's universal law of gravitation, the

*In order to provide a compact notation, the convention will be adopted throughout the discussion that an arbitrary reference frame $\mathcal{A}$ will be taken to be equivalent to a basis $\left\{\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z\right\}$ that is fixed in $\mathcal{A}$. In other words, the notation $\mathcal{A}$ has the same meaning as the basis $\left\{\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z\right\}$, where $\left\{\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z\right\}$ is fixed in reference frame $\mathcal{A}$.

force exerted by the planet on the spacecraft, denoted $\mathbf{F}_{SP}$, is given as

$$\mathbf{F}_{SP} = -\frac{GmM}{\|\mathbf{r}\|^3}\mathbf{r} = -\frac{GmM}{r^3}\mathbf{r}. \tag{1.2}$$

In addition, from the strong form of Newton's third law, the force exerted by the spacecraft on the planet, denoted $\mathbf{F}_{PS}$, is given as

$$\mathbf{F}_{PS} = -\mathbf{F}_{SP} = \frac{GmM}{r^3}\mathbf{r}. \tag{1.3}$$

Applying Newton's second law to both the spacecraft and the planet gives

$$\begin{aligned} \mathbf{F}_{SP} &= m\,{}^{\mathcal{I}}\mathbf{a}_S, && (1.4)\\ \mathbf{F}_{PS} &= M\,{}^{\mathcal{I}}\mathbf{a}_P. && (1.5) \end{aligned}$$

Substituting the models for the forces $\mathbf{F}_{SP}$ and $\mathbf{F}_{PS}$ into Eqs. (1.4) and (1.5) gives

$$\begin{aligned} -\frac{GmM}{r^3}\mathbf{r} &= m\,{}^{\mathcal{I}}\mathbf{a}_S, && (1.6)\\ \frac{GmM}{r^3}\mathbf{r} &= M\,{}^{\mathcal{I}}\mathbf{a}_P. && (1.7) \end{aligned}$$

Suppose now that Eqs. (1.6) and (1.7) are divided by m and M, respectively. Then

$$\begin{aligned} {}^{\mathcal{I}}\mathbf{a}_S &= -\frac{GM}{r^3}\mathbf{r}, && (1.8)\\ {}^{\mathcal{I}}\mathbf{a}_P &= \frac{Gm}{r^3}\mathbf{r}. && (1.9) \end{aligned}$$

Subtracting Eq. (1.9) from (1.8) gives

$${}^{\mathcal{I}}\mathbf{a}_S - {}^{\mathcal{I}}\mathbf{a}_P = {}^{\mathcal{I}}\mathbf{a}_{S/P} = {}^{\mathcal{I}}\mathbf{a} = -\frac{G(M+m)}{r^3}\mathbf{r}, \tag{1.10}$$

where the inertial acceleration of the spacecraft relative to the planet has been denoted ${}^{\mathcal{I}}\mathbf{a}$, that is,

$${}^{\mathcal{I}}\mathbf{a} = {}^{\mathcal{I}}\mathbf{a}_S - {}^{\mathcal{I}}\mathbf{a}_P. \tag{1.11}$$

Equation (1.10) then simplifies to

$${}^{\mathcal{I}}\mathbf{a} = -\frac{G(M+m)}{r^3}\mathbf{r}. \tag{1.12}$$

Next, assume that the planet is significantly more massive than the spacecraft, that is, $M \gg m$. As a result, $M + m \approx M$ and Eq. (1.12) further simplifies to

$${}^{\mathcal{I}}\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}. \tag{1.13}$$

Finally, let $\mu = GM$. Then Eq. (1.13) can be written as

$${}^{\mathcal{I}}\mathbf{a} = -\frac{\mu}{r^3}\mathbf{r}. \tag{1.14}$$

The quantity $\mu = GM$ is called the *gravitational parameter* of the planet (and, because μ depends upon the mass M of the planet, μ is different for every planet). Now, because

$M \gg m$, it is reasonable to approximate point P (that is, the location of the planet) to be an inertially fixed point. In order to facilitate the discussion that follows, for simplicity assume that point P (the location of the planet) and point O are coincident and that from this point forth point O will be the inertially fixed point from which all distances are measured. As a result, ${}^{\mathcal{I}}\mathbf{a}$ is then approximated as the inertial acceleration of the spacecraft (because, as stated, the planet is now considered an inertially fixed point). Equation 1.14 can be rearranged to obtain

$$\boxed{{}^{\mathcal{I}}\mathbf{a} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}.} \tag{1.15}$$

Equation (1.15) is called the *two-body differential equation* and describes the motion of a spacecraft of mass m relative to a planet of mass M, where the planet is signficantly more massive than the spacecraft.

1.3 Solution of Two-Body Differential Equation

In this section the solution of the two-body differential equation is now derived. The solution to the two-body differential equation will be obtained by deriving two constants of integration and using these two constants of integration to obtain the solution. First, define the quantity ${}^{\mathcal{I}}\mathbf{h}$ as

$${}^{\mathcal{I}}\mathbf{h} \equiv {}^{\mathcal{I}}\mathbf{h}_O = (\mathbf{r} - \mathbf{r}_O) \times ({}^{\mathcal{I}}\mathbf{v} - {}^{\mathcal{I}}\mathbf{v}_O). \tag{1.16}$$

Consistent with the earlier assumption that O is the point from which all distances are measured and is fixed in $\mathcal{I}$, Eq. (1.16) reduces to

$$\boxed{{}^{\mathcal{I}}\mathbf{h} = \mathbf{r} \times {}^{\mathcal{I}}\mathbf{v}.} \tag{1.17}$$

The quantity ${}^{\mathcal{I}}\mathbf{h}$ is called the *specific angular momentum* of the spacecraft relative to the planet. Computing the rate of change of ${}^{\mathcal{I}}\mathbf{h}$ relative to the inertial reference frame $\mathcal{I}$ gives

$$\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{h}\right) = \frac{{}^{\mathcal{I}}d}{dt}\left(\mathbf{r} \times {}^{\mathcal{I}}\mathbf{v}\right) = \frac{{}^{\mathcal{I}}d\mathbf{r}}{dt} \times {}^{\mathcal{I}}\mathbf{v} + \mathbf{r} \times \frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\right). \tag{1.18}$$

Now it is noted that

$${}^{\mathcal{I}}\mathbf{v} = \frac{{}^{\mathcal{I}}d\mathbf{r}}{dt}, \quad {}^{\mathcal{I}}\mathbf{a} = \frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\right). \tag{1.19}$$

Substituting the expressions in Eq. (1.19) into Eq. (1.18) gives

$$\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{h}\right) = {}^{\mathcal{I}}\mathbf{v} \times {}^{\mathcal{I}}\mathbf{v} + \mathbf{r} \times {}^{\mathcal{I}}\mathbf{a}. \tag{1.20}$$

Then, using the fact that ${}^{\mathcal{I}}\mathbf{v} \times {}^{\mathcal{I}}\mathbf{v} = \mathbf{0}$ together with the expression for ${}^{\mathcal{I}}\mathbf{a}$ from Eq. (1.14), the rate of change of ${}^{\mathcal{I}}\mathbf{h}$ in Eq. (1.20) can be written as

$$\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{h}\right) = \mathbf{r} \times \left(-\frac{\mu}{r^3}\mathbf{r}\right) = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{r} = \mathbf{0}. \tag{1.21}$$

where it is noted that the quantity $-\mu/r^3$ is a scalar and can be factored out of the last expression in Eq. (1.21). Equation (1.21) shows that the rate of change of ${}^{\mathcal{I}}\mathbf{h}$ is zero which implies that ${}^{\mathcal{I}}\mathbf{h}$ is fixed in the inertial reference frame $\mathcal{I}$ and further implies that the magnitude of ${}^{\mathcal{I}}\mathbf{h}$ is constant, that is,

$$\|{}^{\mathcal{I}}\mathbf{h}\| \equiv h = \text{constant}. \tag{1.22}$$

Now, not only is ${}^{\mathcal{I}}\mathbf{h}$ a constant of the motion for the two-body system, but also because ${}^{\mathcal{I}}\mathbf{h}$ is formed by taking the vector product of $\mathbf{r}$ with ${}^{\mathcal{I}}\mathbf{v}$, it must be the case that ${}^{\mathcal{I}}\mathbf{h}$ is orthognonal to the plane in which the solution of the two-body differential equation lies, that is, ${}^{\mathcal{I}}\mathbf{h}\cdot\mathbf{r} = 0$ and ${}^{\mathcal{I}}\mathbf{h}\cdot{}^{\mathcal{I}}\mathbf{v} = 0$.

Next, the specific angular momentum ${}^{\mathcal{I}}\mathbf{h}$ together with the two-body differential equation of Eq. (1.15) can be used to derive a second constant of integration. Specifically, taking the vector product of Eq. (1.15) on the left with ${}^{\mathcal{I}}\mathbf{h}$ gives

$${}^{\mathcal{I}}\mathbf{a}\times{}^{\mathcal{I}}\mathbf{h} + \frac{\mu}{r^3}\mathbf{r}\times{}^{\mathcal{I}}\mathbf{h} = \mathbf{0}. \tag{1.23}$$

Now because ${}^{\mathcal{I}}\mathbf{h}$ is fixed in $\mathcal{I}$, it is the case that

$${}^{\mathcal{I}}\mathbf{a}\times{}^{\mathcal{I}}\mathbf{h} = \frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}\right). \tag{1.24}$$

Next, from the definition of the specific angular momentum given in Eq. (1.17) it is noted that

$$\mathbf{r}\times{}^{\mathcal{I}}\mathbf{h} = \mathbf{r}\times\left(\mathbf{r}\times{}^{\mathcal{I}}\mathbf{v}\right). \tag{1.25}$$

Then, using the vector triple product identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, Eq. (1.25) can be re-written as

$$\mathbf{r}\times{}^{\mathcal{I}}\mathbf{h} = (\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v})\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\,{}^{\mathcal{I}}\mathbf{v}. \tag{1.26}$$

Therefore, the second term in Eq. (1.23) can be written as

$$\frac{\mu}{r^3}\mathbf{r}\times{}^{\mathcal{I}}\mathbf{h} = \frac{\mu}{r^3}\left[(\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v})\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\,{}^{\mathcal{I}}\mathbf{v}\right] \tag{1.27}$$

Note, however, that

$$\begin{aligned}
\mu\frac{{}^{\mathcal{I}}d}{dt}\left(\frac{\mathbf{r}}{r}\right) &= \mu\frac{{}^{\mathcal{I}}d}{dt}\left(r^{-1}\mathbf{r}\right) \\
&= \mu\frac{{}^{\mathcal{I}}d}{dt}\left([\mathbf{r}\cdot\mathbf{r}]^{-1/2}\mathbf{r}\right) \\
&= \mu\left(-\tfrac{1}{2}[\mathbf{r}\cdot\mathbf{r}]^{-3/2}\left(\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v} + {}^{\mathcal{I}}\mathbf{v}\cdot\mathbf{r}\right)\mathbf{r} + [\mathbf{r}\cdot\mathbf{r}]^{-1/2}\,{}^{\mathcal{I}}\mathbf{v}\right) \\
&= \mu\left(-[\mathbf{r}\cdot\mathbf{r}]^{-3/2}\left(\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v}\right)\mathbf{r} + [\mathbf{r}\cdot\mathbf{r}]^{-1/2}\,{}^{\mathcal{I}}\mathbf{v}\right) \\
&= \mu\left(-\frac{[\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v}]}{r^3}\mathbf{r} + \frac{{}^{\mathcal{I}}\mathbf{v}}{r}\right) \\
&= -\frac{\mu}{r^3}\left([\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v}]\mathbf{r} - r^2\,{}^{\mathcal{I}}\mathbf{v}\right) \\
&= -\frac{\mu}{r^3}\left([\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v}]\mathbf{r} - [\mathbf{r}\cdot\mathbf{r}]\,{}^{\mathcal{I}}\mathbf{v}\right).
\end{aligned} \tag{1.28}$$

Therefore,

$$\frac{\mu}{r^3}\left(\left[\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v}\right]\mathbf{r}-[\mathbf{r}\cdot\mathbf{r}]\,{}^{\mathcal{I}}\mathbf{v}\right)=-\mu\frac{{}^{\mathcal{I}}d}{dt}\left(\frac{\mathbf{r}}{r}\right)=\frac{\mu}{r^3}\mathbf{r}\times{}^{\mathcal{I}}\mathbf{h}. \tag{1.29}$$

Substituting the result of Eq. (1.29) into (1.23) gives

$${}^{\mathcal{I}}\mathbf{a}\times{}^{\mathcal{I}}\mathbf{h}+\frac{\mu}{r^3}\mathbf{r}\times{}^{\mathcal{I}}\mathbf{h}=\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}\right)-\mu\frac{{}^{\mathcal{I}}d}{dt}\left(\frac{\mathbf{r}}{r}\right)=\mathbf{0}. \tag{1.30}$$

The two terms in Eq. (1.30) can be combined to give

$${}^{\mathcal{I}}\mathbf{a}\times{}^{\mathcal{I}}\mathbf{h}+\frac{\mu}{r^3}\mathbf{r}\times{}^{\mathcal{I}}\mathbf{h}=\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}-\mu\frac{\mathbf{r}}{r}\right)=\mathbf{0}. \tag{1.31}$$

Consequently,

$${}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}-\mu\frac{\mathbf{r}}{r}=\mathbf{C}, \tag{1.32}$$

where $\mathbf{C}$ is fixed in the inertial reference frame $\mathcal{I}$. The inertially fixed vector $\mathbf{C}$ is called the *Laplace vector* and is fixed in the inertial reference frame $\mathcal{I}$. A more commonly used form of the Laplace vector is obtained by dividing Eq. (1.32) by μ as

$$\boxed{\frac{{}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}}{\mu}-\frac{\mathbf{r}}{r}=\mathbf{e}.} \tag{1.33}$$

The vector $\mathbf{e}$ is called the *eccentricity vector* and, similar to the Laplace vector, is fixed in the inertial reference frame $\mathcal{I}$. It is noted that the eccentricity vector is a constant of the motion (actually, it comprises three scalar constants of the motion because $\mathbf{e}$ is itself a vector). Now, it is seen that

$$\mathbf{e}\cdot{}^{\mathcal{I}}\mathbf{h}=\left[\frac{{}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}}{\mu}-\frac{\mathbf{r}}{r}\right]\cdot{}^{\mathcal{I}}\mathbf{h}=\frac{{}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}}{\mu}\cdot{}^{\mathcal{I}}\mathbf{h}-\frac{\mathbf{r}}{r}\cdot{}^{\mathcal{I}}\mathbf{h}=\mathbf{0}. \tag{1.34}$$

Consequently, the eccentricity vector $\mathbf{e}$ is orthogonal to ${}^{\mathcal{I}}\mathbf{h}$. Moreover, because ${}^{\mathcal{I}}\mathbf{h}$ is orthogonal to the plane of the solution of the two-body differential equation it must be the case that $\mathbf{e}$ lies in the plane of the solution of the two-body differential equation given in Eq. (1.15).

The two orthogonal constant vectors ${}^{\mathcal{I}}\mathbf{h}$ and $\mathbf{e}$ can now be used to derive the solution of the two-body differential equation given in Eq. (1.15). First, recall that the eccentricity vector $\mathbf{e}$ lies in the plane of the solution of Eq. (1.15). Then because $\mathbf{e}$ is fixed in the inertial reference frame $\mathcal{I}$, the magnitude of $\mathbf{e}$ is constant, that is,

$$e=\|\mathbf{e}\|=\text{constant} \tag{1.35}$$

The quantity e is called the *eccentricity* of the orbit. Then, taking the scalar product of $\mathbf{r}$ with $\mathbf{e}$ gives

$$\mathbf{e}\cdot\mathbf{r}=er\cos\nu, \tag{1.36}$$

where ν is the angle between $\mathbf{e}$ and $\mathbf{r}$. Returning now to the expression for $\mathbf{e}$ as given in Eq. (1.33), it is seen that

$$\begin{aligned}\mathbf{e}\cdot\mathbf{r}&=\frac{{}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}}{\mu}\cdot\mathbf{r}-\frac{\mathbf{r}}{r}\cdot\mathbf{r}=\frac{{}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}}{\mu}\cdot\mathbf{r}-\frac{r^2}{r}\\&=\frac{{}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}}{\mu}\cdot\mathbf{r}-r=\frac{\left({}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}\right)\cdot\mathbf{r}}{\mu}-r.\end{aligned} \tag{1.37}$$

Now it is noted from the scalar triple product that $(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ which implies that

$$\left({}^{\mathcal{I}}\mathbf{v}\times{}^{\mathcal{I}}\mathbf{h}\right)\cdot\mathbf{r} = \left(\mathbf{r}\times{}^{\mathcal{I}}\mathbf{v}\right)\cdot{}^{\mathcal{I}}\mathbf{h} = {}^{\mathcal{I}}\mathbf{h}\cdot{}^{\mathcal{I}}\mathbf{h} = h^2. \tag{1.38}$$

Equation 1.37 can then be re-written as

$$\mathbf{e}\cdot\mathbf{r} = \frac{h^2}{\mu} - r. \tag{1.39}$$

Setting the expressions in Eqs. (1.36) and (1.39) equal to one another gives

$$er\cos\nu = \frac{h^2}{\mu} - r. \tag{1.40}$$

Rearranging Eq. (1.40) gives

$$r(1+e\cos\nu) = \frac{h^2}{\mu}. \tag{1.41}$$

Solving Eq. (1.41) for r gives

$$\boxed{r = \frac{h^2/\mu}{1+e\cos\nu}.} \tag{1.42}$$

Equation (1.42) defines a conic section where the form of the conic section depends upon the value of e. Now, it is noted that r achieves its minimum when $\nu = 0$ and that when ν is zero the position vector $\mathbf{r}$ is aligned with $\mathbf{e}$. The points at which r is at its minimum and maximum are called, respectively, the *periapsis* and *apoapsis*. Thus, it is seen that when ν is zero the spacecraft is at periapsis which implies that $\mathbf{e}$ must lie along the direction from the planet to the periapsis. The angle ν is called the *true anomaly* of the orbit. Equation (1.42) is called the *orbit equation* and defines the solution of the two-body differential equation of Eq. (1.15) for the radius r (where it is noted again that $r = \|\mathbf{r}\|$ is the distance from the planet to the spacecraft) in terms of the true anomaly. A convenient way to visualize the geometry of the orbit equation is shown in Fig. 1.2. Specifically, Fig. 1.2 shows that the direction of the specific angular momentum ${}^{\mathcal{I}}\mathbf{h}$ in Eq. (1.17) is orthogonal to the orbit plane and the orbit plane is fixed in the inertial reference frame $\mathcal{I}$. Moreover, the position and inertial velocity of the spacecraft, $\mathbf{r}$ and ${}^{\mathcal{I}}\mathbf{v}$, both lie in the orbit plane.

1.4 Properties of the Orbit Equation

Several properties of the orbit equation are now derived. In order to derive these important quantities it is useful to first define the quantity

$$\boxed{p = \frac{h^2}{\mu}.} \tag{1.43}$$

which implies that

$$h = \sqrt{\mu p} \tag{1.44}$$

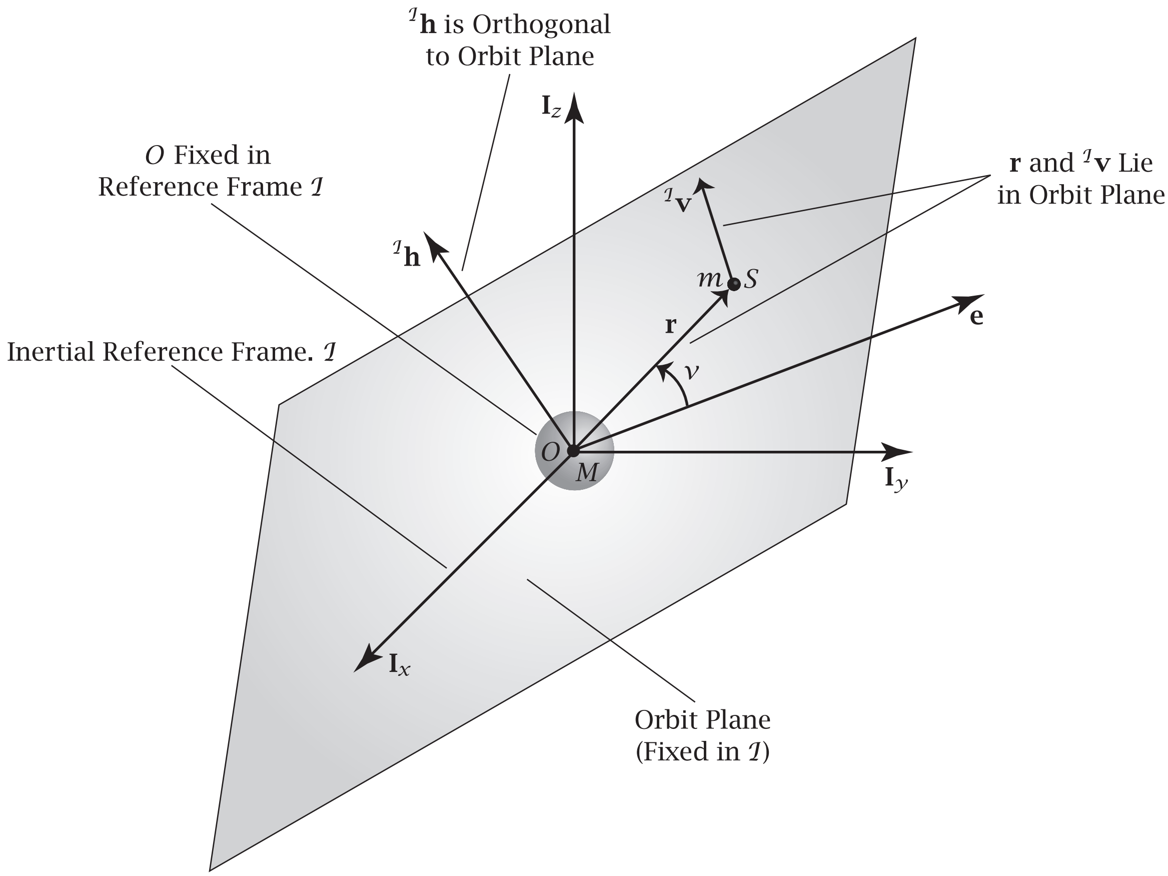


Figure 1.2 Schematic showing the plane in which the solution of the two-body differential equation of Eq. (1.15) lies along with the specific angular momentum that lies orthogonal to the plane of the solution.

The quantity p in Eq. (1.43) is called the *semi-latus rectum* or the *parameter*. In terms of the semi-latus rectum, the orbit equation of Eq. (1.42) can be written in terms as

$$\boxed{r = \frac{p}{1 + e\cos\nu}.} \tag{1.45}$$

It is the form of the orbit equation shown in Eq. (1.45) that will be used in all subsequent derivations.

1.4.1 Periapsis and Apoapsis Radii

Recall from earlier that the periapsis and apoapsis were defined as the points where the spacecraft is closest to and furthest from from the planet, respectively. The periapsis occurs when $\nu = 0$ while the apoapsis occurs when $\nu = \pi$. Suppose that the periapsis and apoapsis radii are denoted as r_p and r_a, respectively. Then, given the locations of periapsis and apoapsis, the quantities r_p and r_a are given from Eq. (1.45), respectively, as

$$\boxed{r_p = \frac{p}{1+e}} \tag{1.46}$$

and

$$\boxed{r_a = \frac{p}{1-e}.} \tag{1.47}$$

Next, the sum of the periapsis and apoapsis radii is called the *major-axis* and is denoted $2a$, that is,

$$2a = r_p + r_a. \tag{1.48}$$

The quantity of interest for an orbit is what is known as the *semi-major axis*, a, and is given as

$$\boxed{a = \frac{r_p + r_a}{2}.} \tag{1.49}$$

Equations (1.46) and (1.47) can then be used to relate the semi-latus rectum, the eccentricity, and the semi-major axis as follows. First, adding Eqs. (1.46) and (1.47) and using the definition in Eq. (1.48) gives

$$r_p + r_a = 2a = \frac{p}{1+e} + \frac{p}{1-e} = \frac{2p}{1-e^2} \tag{1.50}$$

Equation (1.50) then implies that the semi-major axis is related to the semi-latus rectum and the eccentricity as

$$\boxed{a = \frac{p}{1-e^2}.} \tag{1.51}$$

Alternatively, rearranging Eq. (1.51), the semi-latus rectum is related to the semi-major axis and the eccentricity as

$$\boxed{p = a(1-e^2).} \tag{1.52}$$

Then, substituting p from Eq. (1.52) into (1.46) and (1.47), the periapsis and apoapsis radii are given as

$$r_p = \frac{a(1-e^2)}{1+e} \tag{1.53}$$

and

$$r_a = \frac{a(1-e^2)}{1-e}. \tag{1.54}$$

Simplifying Eqs. (1.53) and (1.54), the periapsis and apoapsis radii are given as

$$\boxed{r_p = a(1-e)} \tag{1.55}$$

and

$$\boxed{r_a = a(1+e).} \tag{1.56}$$

Next, Eqs. (1.46) and (1.47) can be used together to solve for e in terms of r_p and r_a. First, subtracting Eq. (1.46) from (1.47) gives

$$r_a - r_p = \frac{p}{1-e} - \frac{p}{1+e} = \frac{p(1+e) - p(1-e)}{(1+e)(1-e)} = \frac{2pe}{1-e^2}. \tag{1.57}$$

Then, dividing Eq. (1.57) by (1.50) gives

$$\boxed{e = \frac{r_a - r_p}{r_a + r_p}.} \tag{1.58}$$

Now, for an ellipse (the case where $0 < e < 1$), the distance between the two foci, defined as $2c$, is given as

$$2c = r_a - r_p. \tag{1.59}$$

Substituting the result of Eq. (1.59) into (1.58) gives

$$e = \frac{c}{a} \tag{1.60}$$

which implies that

$$\boxed{c = ae} \tag{1.61}$$

The *semi-minor axis* is then related to the semi-major axis and the distance between the two foci as

$$b^2 = a^2 - c^2 = a^2 - a^2e^2 = a^2(1 - e^2) \tag{1.62}$$

which implies that

$$\boxed{b = a\sqrt{1 - e^2}.} \tag{1.63}$$

1.4.2 Specific Mechanical Energy

Suppose we define the quantities $\mathfrak{T}$ and $\mathfrak{U}$ as follows:

$$\mathfrak{T} = \frac{1}{2}{}^{\mathcal{I}}\mathbf{v} \cdot {}^{\mathcal{I}}\mathbf{v}, \tag{1.64}$$

$$\mathfrak{U} = -\frac{\mu}{r}. \tag{1.65}$$

The quantities $\mathfrak{T}$ and $\mathfrak{U}$ are defined, respectively, as the *specific kinetic energy* and the *specific potential energy*, respectively, relative to the inertial reference frame $\mathcal{I}$. Next, let $\mathfrak{E}$ be the sum of $\mathfrak{T}$ and $\mathfrak{U}$, that is,

$$\mathfrak{E} = \mathfrak{T} + \mathfrak{U} = \frac{1}{2}{}^{\mathcal{I}}\mathbf{v} \cdot {}^{\mathcal{I}}\mathbf{v} - \frac{\mu}{r} \tag{1.66}$$

The quantity $\mathfrak{E}$ is called the *specific mechanical energy* of the spacecraft relative to the inertial reference frame. The rate of change of the specific mechanical energy is then given as

$$\frac{d\mathfrak{E}}{dt} = \frac{d}{dt}\left(\frac{1}{2}{}^{\mathcal{I}}\mathbf{v} \cdot {}^{\mathcal{I}}\mathbf{v} - \frac{\mu}{r}\right) = \frac{d\mathfrak{T}}{dt} + \frac{d\mathfrak{U}}{dt}. \tag{1.67}$$

Now because $\mathfrak{E}$ is a scalar, the rate of change of $\mathfrak{E}$ can be taken in any reference frame. Suppose arbitrarily that the rate of change of $\mathfrak{E}$ is taken in the inertial reference frame $\mathcal{I}$. Then the rate of change of $\mathfrak{E}$ is given as

$$\frac{d\mathfrak{E}}{dt} = \frac{d\mathfrak{T}}{dt} + \frac{d\mathfrak{U}}{dt} = \frac{{}^{\mathcal{I}}d}{dt}\left(\frac{1}{2}{}^{\mathcal{I}}\mathbf{v} \cdot {}^{\mathcal{I}}\mathbf{v}\right) - \frac{{}^{\mathcal{I}}d}{dt}\left(\frac{\mu}{r}\right). \tag{1.68}$$

First, the rate of change of the specific kinetic energy in the inertial reference frame is given as

$$\frac{d\mathfrak{T}}{dt} = \frac{1}{2}\left({}^{\mathcal{I}}\mathbf{a}\cdot{}^{\mathcal{I}}\mathbf{v} + {}^{\mathcal{I}}\mathbf{v}\cdot{}^{\mathcal{I}}\mathbf{a}\right) = \frac{1}{2}\left(2\,{}^{\mathcal{I}}\mathbf{v}\cdot{}^{\mathcal{I}}\mathbf{a}\right) = {}^{\mathcal{I}}\mathbf{a}\cdot{}^{\mathcal{I}}\mathbf{v}, \tag{1.69}$$

where it is noted that ${}^{\mathcal{I}}\mathbf{a} = {}^{\mathcal{I}}d({}^{\mathcal{I}}\mathbf{v})/dt$. Then, from the two-body differential equation of Eq. (1.15), ${}^{\mathcal{I}}\mathbf{a} = -\mu\mathbf{r}/r^3$ from which $d\mathfrak{T}/dt$ can be re-written as

$$\frac{d\mathfrak{T}}{dt} = -\frac{\mu}{r^3}\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v}. \tag{1.70}$$

Next, noting that $r = [\mathbf{r}\cdot\mathbf{r}]^{1/2}$, the rate of change of the specific potential energy in the inertial frame is given as

$$\begin{aligned}\frac{d\mathfrak{U}}{dt} &= \frac{d}{dt}\left(-\frac{\mu}{r}\right) = \frac{d}{dt}\left(-\mu[\mathbf{r}\cdot\mathbf{r}]^{-1/2}\right) = \frac{1}{2}\mu[\mathbf{r}\cdot\mathbf{r}]^{-3/2}\left({}^{\mathcal{I}}\mathbf{v}\cdot\mathbf{r} + {}^{\mathcal{I}}\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v}\right)\\ &= \frac{\mu}{[\mathbf{r}\cdot\mathbf{r}]^{3/2}}\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v} = \frac{\mu}{r^3}\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v},\end{aligned} \tag{1.71}$$

where it is noted that ${}^{\mathcal{I}}\mathbf{v} = {}^{\mathcal{I}}d\mathbf{r}/dt$ and $[\mathbf{r}\cdot\mathbf{r}]^{3/2} = r^3$. Substituting the expressions for $d\mathfrak{T}/dt$ and $d\mathfrak{U}/dt$ from Eqs. (1.70) and (1.71), respectively, the rate of change of the specific mechanical energy is given as

$$\frac{d\mathfrak{E}}{dt} = -\frac{\mu}{r^3}\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v} + \frac{\mu}{r^3}\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v} = 0. \tag{1.72}$$

Equation (1.72) states that the rate of change of the specific mechanical energy of the spacecraft is zero which implies that

$$\mathfrak{E} = \text{constant}. \tag{1.73}$$

Then, because $\mathfrak{E}$ is constant, this constant can be obtained at any convenient point on the orbit. Thus, arbitrarily choose to evaluate this constant using the conditions at periapsis, where the conditions at periapsis are evaluated conveniently using a so called *perifocal basis* $\left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$ defined as follows:

$$\begin{aligned}\mathbf{p}_x &= \frac{\mathbf{e}}{\|\mathbf{e}\|} = \frac{\mathbf{e}}{e},\\ \mathbf{p}_z &= \frac{{}^{\mathcal{I}}\mathbf{h}}{\|{}^{\mathcal{I}}\mathbf{h}\|} = \frac{{}^{\mathcal{I}}\mathbf{h}}{h},\\ \mathbf{p}_y &= \mathbf{p}_z \times \mathbf{p}_x.\end{aligned} \tag{1.74}$$

Because the vectors $\mathbf{e}$ and ${}^{\mathcal{I}}\mathbf{h}$ are fixed in the inertial reference frame $\mathcal{I}$, it is seen that the perifocal basis $\left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$ is fixed in $\mathcal{I}$. Next, let $\{\mathbf{u}_r, \mathbf{u}_\nu, \mathbf{u}_z\}$ be a basis that is fixed in a reference frame $\mathcal{U}$ and defined as

$$\begin{aligned}\mathbf{u}_r &= \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{\mathbf{r}}{r},\\ \mathbf{u}_z &= \mathbf{p}_z,\\ \mathbf{u}_\nu &= \mathbf{u}_z \times \mathbf{u}_r.\end{aligned} \tag{1.75}$$

Figure 1.3 provides a schematic of the bases $\left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$ and $\{\mathbf{u}_r, \mathbf{u}_\nu, \mathbf{u}_z\}$. As seen from Fig. 1.3, ν is the angle from $\mathbf{p}_x$ to $\mathbf{u}_r$. Consequently, reference frame $\mathcal{U}$ (in which the basis $\{\mathbf{u}_r, \mathbf{u}_\nu, \mathbf{u}_z\}$ is fixed) rotates with an angular rate $\dot{\nu}$ about the $\mathbf{u}_z = \mathbf{p}_z$–direction relative to reference frame $\mathcal{I}$ (in which the basis $\left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$ is fixed) which implies that the angular velocity of reference frame $\mathcal{U}$ relative to the inertial reference frame $\mathcal{I}$ is

$$^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{U}} = \dot{\nu}\mathbf{u}_z. \tag{1.76}$$

In terms of the basis $\{\mathbf{u}_r, \mathbf{u}_\nu, \mathbf{u}_z\}$, the position of the spacecraft relative to the planet

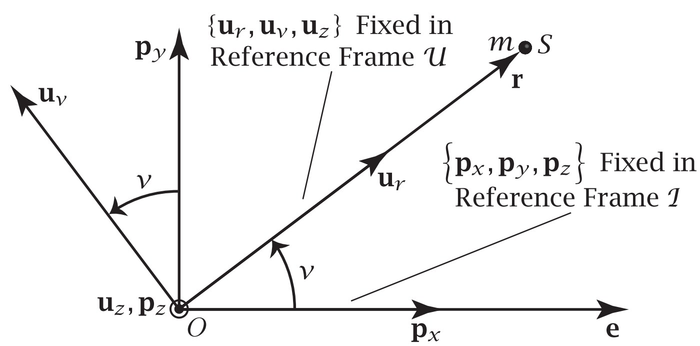


Figure 1.3 Two-dimensional projection showing the bases $\left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$ and $\{\mathbf{u}_r, \mathbf{u}_\nu, \mathbf{u}_z\}$ that lie in the orbit plane for the two-body problem.

is given as

$$\mathbf{r} = r\mathbf{u}_r. \tag{1.77}$$

Applying the transport theorem to the position $\mathbf{r}$ given in Eq. (1.77), the velocity of the spacecraft as viewed by an observer in the inertial reference frame $\mathcal{I}$ is given as

$$^{\mathcal{I}}\mathbf{v} = \frac{^{\mathcal{I}}d\mathbf{r}}{dt} = \frac{^{\mathcal{U}}d\mathbf{r}}{dt} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{U}} \times \mathbf{r} = \dot{r}\mathbf{u}_r + \dot{\nu}\mathbf{u}_z \times r\mathbf{u}_r = \dot{r}\mathbf{u}_r + r\dot{\nu}\mathbf{u}_\nu \tag{1.78}$$

Using the expression for $^{\mathcal{I}}\mathbf{v}$ given Eq. (1.78), the specific kinetic energy is given as

$$\mathfrak{T} = \tfrac{1}{2}{}^{\mathcal{I}}\mathbf{v} \cdot {}^{\mathcal{I}}\mathbf{v} = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\nu}^2) \tag{1.79}$$

Now it is noted that

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\nu}\frac{d\nu}{dt} \tag{1.80}$$

Taking the derivative of r in Eq. (1.42) with respect to ν gives

$$\frac{dr}{d\nu} = \frac{pe\sin\nu}{(1+e\cos\nu)^2} = \frac{p^2 e\sin\nu}{p(1+e\cos\nu)^2} = \frac{r^2 e\sin\nu}{p}. \tag{1.81}$$

Furthermore, the specific angular momentum $^{\mathcal{I}}\mathbf{h}$ defined in Eq. (1.17) can be expressed in terms of the basis $\{\mathbf{u}_r, \mathbf{u}_\nu, \mathbf{u}_z\}$ as

$$^{\mathcal{I}}\mathbf{h} = \mathbf{r} \times {}^{\mathcal{I}}\mathbf{v} = r\mathbf{u}_r \times (\dot{r}\mathbf{u}_r + r\dot{\nu}\mathbf{u}_\nu) = r^2\dot{\nu}\mathbf{u}_z = h\mathbf{u}_z. \tag{1.82}$$

which implies that

$$\boxed{\dot{\nu} = \frac{h}{r^2}} \tag{1.83}$$

Therefore, the rate of change of r is obtained as

$$\dot{r} = \frac{r^2 e \sin \nu}{p} \frac{h}{r^2} = \frac{he \sin \nu}{p}. \tag{1.84}$$

The value of $\dot{r}$ at periapsis (that is, the value of $\dot{r}$ when $\nu = 0$), is then given as

$$\dot{r}(\nu = 0) = \left[\frac{dr}{dt}\right]_{\nu=0} = \left[\frac{he \sin \nu}{p}\right]_{\nu=0} = 0. \tag{1.85}$$

Consequently, the specific kinetic energy evaluated at the periapsis of the orbit is obtained as

$$\mathfrak{T}(\nu = 0) = \tfrac{1}{2}\left[{}^{\mathcal{I}}\mathbf{v} \cdot {}^{\mathcal{I}}\mathbf{v}\right]_{\nu=0} = \tfrac{1}{2} r_p^2 \left(\frac{h}{r_p^2}\right)^2 = \frac{h^2}{2r_p^2} \tag{1.86}$$

Also, the specific potential energy at periapsis is given as

$$\mathfrak{U}(\nu = 0) = -\frac{\mu}{r_p}. \tag{1.87}$$

Therefore, the specific mechanical energy can be written as

$$\mathfrak{E} = \mathfrak{T} + \mathfrak{U} = \frac{h^2}{2r_p^2} - \frac{\mu}{r_p}. \tag{1.88}$$

Then, noting from Eq. (1.43) that $h^2 = \mu p$, Eq. (1.88) can be written as

$$\mathfrak{E} = \frac{\mu p}{2r_p^2} - \frac{\mu}{r_p} = \frac{\mu}{r_p}\left(\frac{p}{2r_p} - 1\right) = \frac{\mu}{r_p} \frac{p - 2r_p}{2r_p}. \tag{1.89}$$

Then, substituting the results of Eqs. (1.52), and (1.55) into Eq. (1.89) gives

$$\begin{aligned} \mathfrak{E} &= \frac{\mu}{r_p} \frac{a(1-e^2) - 2a(1-e)}{2a(1-e)} = \frac{\mu}{r_p} \frac{(1+e)(1-e) - 2(1-e)}{1-e} \\ &= -\frac{\mu(1-e)}{2r_p} = -\frac{\mu(1-e)}{2a(1-e)} = -\frac{\mu}{2a}. \end{aligned} \tag{1.90}$$

Therefore, the specific mechanical energy reduces to

$$\boxed{\mathfrak{E} = -\frac{\mu}{2a}.} \tag{1.91}$$

Now, Eq. (1.91) can be used to derive the following additional result that is often quite useful. First, note that

$${}^{\mathcal{I}}\mathbf{v} \cdot {}^{\mathcal{I}}\mathbf{v} = v^2, \tag{1.92}$$

where $v = \|{}^{\mathcal{I}}\mathbf{v}\|$ is the inertial speed. Therefore, the specific mechanical energy is given as

$$\mathfrak{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}. \tag{1.93}$$

Solving Eq. (1.93) for v^2 gives

$$\boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right).} \tag{1.94}$$

Equation (1.94) is called the *vis-viva* equation. From the vis-viva equation it can be seen that the speed of the spacecraft at any point on the solution of the two-body differential equation of Eq. (1.15) is given as

$$\boxed{v = \sqrt{\mu}\sqrt{\frac{2}{r} - \frac{1}{a}}.} \tag{1.95}$$

1.4.3 Flight Path Angle

Another important quantity of the orbit is the *flight path angle* denoted γ. First, because the specific angular momentum lies along the direction $\mathbf{u}_z$ where $\mathbf{u}_z = \mathbf{p}_z$ and is orthogonal to the orbit plane,

$$^{\mathcal{I}}\mathbf{h} = \mathbf{r} \times {}^{\mathcal{I}}\mathbf{v} = rv\mathbf{u}_z \sin\phi, \tag{1.96}$$

where r is as given in Eq. (1.45), v is as given in Eq. (1.95), and ϕ is the angle between $\mathbf{r}$ and $^{\mathcal{I}}\mathbf{v}$ and is called the *zenith angle*. Taking the magnitude of Eq. (1.96) gives

$$h = \|{}^{\mathcal{I}}\mathbf{h}\| = \|\mathbf{r} \times {}^{\mathcal{I}}\mathbf{v}\| = rv\sin\phi. \tag{1.97}$$

Now, let

$$\gamma = \frac{\pi}{2} - \phi. \tag{1.98}$$

which implies that

$$\phi = \frac{\pi}{2} - \gamma. \tag{1.99}$$

The angle γ is called the *flight path angle*. Both the zenith angle and the flight path angle are shown in Fig. 1.4, where the directions $\mathbf{u}_r$ and $\mathbf{u}_\nu$ defined via the basis $\{\mathbf{u}_r, \mathbf{u}_\nu, \mathbf{u}_z\}$ in Eq. (1.75) are called, respectively, the *local vertical* and *local horizontal* directions. It is seen that the zenith angle is defined as the angle from the local vertical to the direction of the inertial velocity while the flight path angle is defined as the angle from the local horizontal to the direction of the velocity.

Using Eq. (1.98) together with the fact that $\sin\phi = \sin(\pi/2 - \gamma) = \cos\gamma$, Eq. (1.97) becomes

$$h = rv\cos\gamma. \tag{1.100}$$

Next, taking the scalar product of $\mathbf{r}$ with $^{\mathcal{I}}\mathbf{v}$ gives

$$\mathbf{r} \cdot {}^{\mathcal{I}}\mathbf{v} = \|\mathbf{r}\|\|{}^{\mathcal{I}}\mathbf{v}\|\cos\phi = rv\cos\phi. \tag{1.101}$$

Then, using the identity $\cos\phi = \cos(\pi/2 - \gamma) = \sin\gamma$ gives

$$\mathbf{r} \cdot {}^{\mathcal{I}}\mathbf{v} = rv\sin\gamma. \tag{1.102}$$

Combining the results in Eq. (1.100) and (1.102), the tangent of the flight path angle is given as

$$\tan\gamma = \frac{\mathbf{r} \cdot {}^{\mathcal{I}}\mathbf{v}}{h}. \tag{1.103}$$

Now it is noted that

$$\mathbf{r} \cdot {}^{\mathcal{I}}\mathbf{v} = \frac{1}{2}\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \frac{1}{2}\frac{d}{dt}\left(r^2\right) = r\dot{r} \tag{1.104}$$

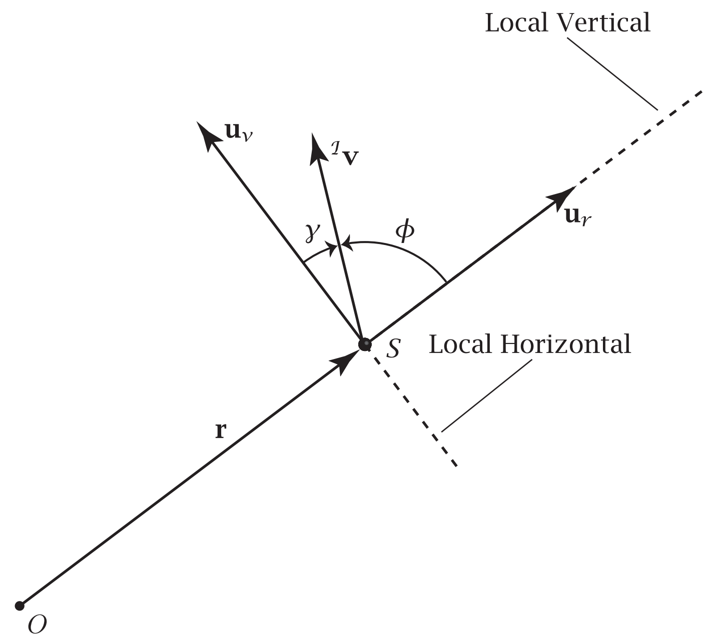


Figure 1.4 Zenith angle, ϕ, and flight path angle, γ, along with the local vertical and local horizontal directions, $\mathbf{u}_r$ and $\mathbf{u}_\nu$.

which implies that

$$\tan\gamma = \frac{r\dot{r}}{h}. \tag{1.105}$$

Then, substituting the result of Eq. (1.84) into (1.105) gives

$$\tan\gamma = \frac{r\dfrac{he\sin\nu}{p}}{h} = \frac{re\sin\nu}{p}. \tag{1.106}$$

Then, substituting Eq. (1.45) into (1.106) gives

$$\boxed{\tan\gamma = \frac{e\sin\nu}{1+e\cos\nu}.} \tag{1.107}$$

Equation (1.107) provides a simple relationship for the tangent of the flightpath angle in terms of the eccentricity and true anomaly. Now, because the zenith angle, ϕ, is the angle between $\mathbf{r}$ and ${}^{\mathcal{I}}\mathbf{v}$, it follows that $\phi \in [0, \pi]$, that is,

$$0 \le \phi \le \pi. \tag{1.108}$$

Then, substituting the expression for the zenith angle in terms of the flight path angle, Eq. (1.108) is given as

$$0 \le \frac{\pi}{2} - \gamma \le \pi. \tag{1.109}$$

which implies that

$$-\frac{\pi}{2} \le \gamma \le \frac{\pi}{2}. \tag{1.110}$$

Therefore, $\gamma \in [-pi/2, \pi/2]$. Therefore, the flight path angle can be computed using the standard inverse tangent as

$$\boxed{\gamma = \tan^{-1}\left(\frac{e\sin\nu}{1+e\cos\nu}\right).} \tag{1.111}$$

Finally, it is noted from Eq. (1.111) that the flight path angle is zero when the true anomaly is either zero or π, that is, the flight path angle is zero when the spacecraft is at either the periapsis or the apoapsis of the orbit.

1.4.4 Period of an Orbit

Another important property of orbit equation is the period of the orbit. The orbital period is derived via Kepler's second law of planetary motion, namely, an equal area is swept out by the orbit in equal time. In order to determine the relationship between the area swept out by the ellipse and the time interval over which this area is swept out, consider an incremental change in the true anomaly as shown in Fig. 1.5. Specifically,

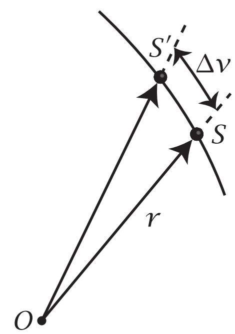


Figure 1.5 Incremental change in true anomaly ν on an orbit defined by Eq. (1.45).

Fig. 1.5 shows the location S of the spacecraft on the orbit when the true anomaly has a value ν and the location S' of the spacecraft when the true anomaly has a value $\nu + \Delta\nu$. For sufficiently small $\Delta\nu$, the area ΔA enclosed by the triangle OSS' is given as

$$\Delta A \approx \frac{1}{2}BH = \frac{1}{2}(r\Delta\nu)r = \frac{1}{2}r^2\Delta\nu, \tag{1.112}$$

where $B = r\Delta\nu$ and $H = r$ are the base and the height, respectively, of the triangle OSS'. Therefore, over a small time increment Δt,

$$\frac{\Delta A}{\Delta t} \approx \frac{1}{2}r^2\frac{\Delta\nu}{\Delta t}. \tag{1.113}$$

In the limit as $\Delta t \to 0$ Eq. (1.113) becomes

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\nu}{dt}. \tag{1.114}$$

Note, from Eq. (1.83) that $h = r^2\dot{\nu}$ which implies that

$$\frac{dA}{dt} = \frac{1}{2}h \tag{1.115}$$

Finally, because h is constant, Eq. (1.115), the rate of change of A is a constant which implies that equal areas are swept out over equal times on an orbit, that is,

$$A(t_2) - A(t_1) = \frac{1}{2}h(t_2 - t_1). \tag{1.116}$$

Equation (1.116) is referred to as *Kepler's second law*. Kepler's second law can now be used to determine the orbital period of the spacecraft. First, rearranging Eq. (1.115) gives

$$dt = \frac{2}{h}dA. \tag{1.117}$$

Now let τ be the orbital period, that is, τ is the time it takes to traverse a true anomaly of 2π. Furthermore, let A be the area swept out by the conic section in time τ. Then, given that h is constant, Eq. (1.117) can be integrated to obtain

$$\tau = \frac{2}{h}A, \tag{1.118}$$

where A represents the area swept out as the spacecraft makes one orbital revolution about the planet. Now note for an ellipse that

$$A = \pi ab \tag{1.119}$$

where a and b are the semi-major axis and semi-minor axis, respectively. Using the result of Eq. (1.62) on page 16 in Eq. (1.119), the area of an ellipse can be written as

$$A = \pi a^2\sqrt{1-e^2}. \tag{1.120}$$

Furthermore, substituting A from Eq. (1.120) into (1.118) gives

$$\tau = \frac{2\pi a^2\sqrt{1-e^2}}{h}. \tag{1.121}$$

Now it is noted that from Eqs. (1.52) and Eq. (1.43) that $1 - e^2 = p/a$ and $h = \sqrt{\mu p}$, respectively, which implies that

$$\tau = \frac{2\pi a^2\sqrt{p/a}}{h} = 2\pi\frac{a^{3/2}\sqrt{p}}{\sqrt{\mu p}} \tag{1.122}$$

Simplifying Eq. (1.122), the orbital period of the spacecraft is given as

$$\boxed{\tau = 2\pi\sqrt{\frac{a^3}{\mu}}.} \tag{1.123}$$

1.5 Types of Orbits

In Section 1.4 several key properties of the orbit equation given in Eq. (1.45) were derived. In this section, these properties will be further specialized to provide the properties of the orbit based on the value of the orbital eccentricity. In particular, the value of the orbital eccentricity states whether the orbit is an ellipse, a parabola, or an hyperbola. Each of these cases is now considered.

1.5.1 Elliptic Orbit: $0 \le e < 1$

An elliptic orbit is one where $0 \le e < 1$ (that is, the eccentricity is greater than or equal to zero and strictly less than unity). The solution of equation given in Eq. (1.45) for $0 \le e < 1$ is an ellipse and is shown in Fig. 1.6. It is seen from Fig. 1.6 that the properties of Eq. (1.45) all hold in their native forms for the case of an elliptic orbit. First, it is seen for an ellipse that $\nu = 0$ and $\nu = \pi$ correspond to the periapsis and apoapsis of the orbit. Consequently, the periapsis and apoapsis radii are given exactly as shown in Eqs. (1.46) and (1.47). Furthermore, it is seen from Eq. (1.93) that the energy on an elliptic orbit is negative, that is,

$$\mathcal{E} = -\frac{\mu}{2a} < 0 \tag{1.124}$$

because $a > 0$ for an elliptic orbit. Finally, the orbital period of an elliptic orbit as given in Eq. (1.123).

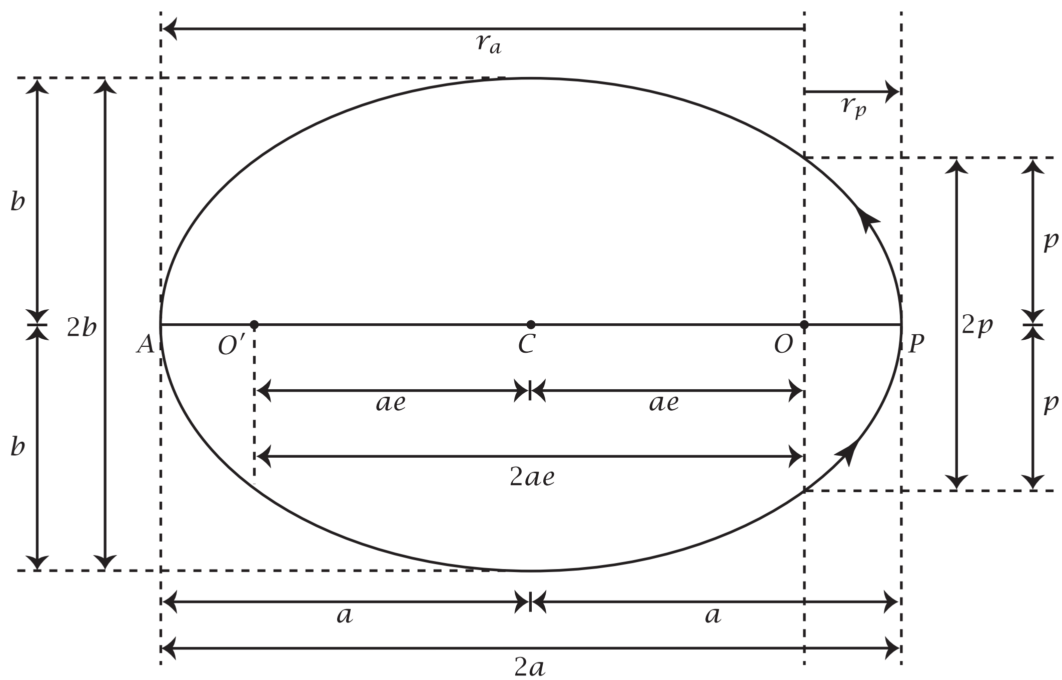


Figure 1.6 Elliptic orbit that arises from the solution of the orbit equation given in Eq. (1.45) for the case where $0 \le e < 1$.

1.5.2 Parabolic Orbit: $e = 1$

A parabolic orbit is one where $e = 1$ (that is, the eccentricity is exactly unity). The solution of equation given in Eq. (1.45) for $e = 1$ is given as

$$r = \frac{p}{1 + \cos \nu} \tag{1.125}$$

and the parabolic trajectory corresponding to Eq. (1.125) is shown in Fig. 1.7. Substituting $\nu = 0$ into Eq. (1.125), it is seen that the periapsis radius is given as

$$r_p = \frac{p}{1 + \cos 0} = \frac{p}{2}. \tag{1.126}$$

Furthermore, substituting $\nu = \pi$ into Eq. (1.125), it is seen that the apoapsis radius is given as

$$r_a = \frac{p}{1 + \cos \pi} = \infty \tag{1.127}$$

which implies that the apoapsis radius for a parabola is ∞. Because the apoapsis radius is infinite, the semi-major axis for a parabolic orbit is

$$a = \frac{r_p + r_a}{2} = \frac{p/2 + \infty}{2} = \infty \tag{1.128}$$

which implies that the energy on a parabolic orbit is

$$\mathcal{E} = -\frac{\mu}{2a} = -\frac{\mu}{2 \cdot \infty} = 0. \tag{1.129}$$

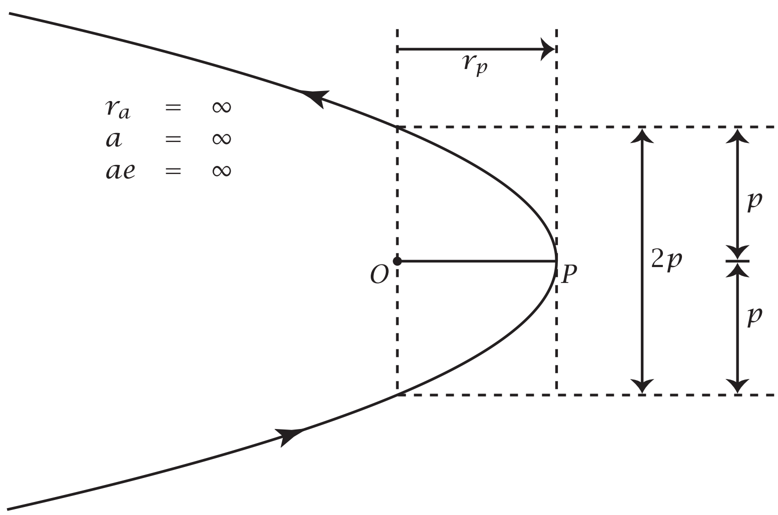


Figure 1.7 Parabolic orbit that arises from the solution of the orbit equation given in Eq. (1.45) for the case where $e = 1$.

1.5.3 Hyperbolic Orbit: $e > 1$

A hyperbolic orbit is one where $e > 1$ (that is, the eccentricity is strictly greater than unity). The orbit equation for for $e > 1$ is the same as that given in Eq. (1.45), that is,

$$r = \frac{p}{1 + e \cos \nu} \tag{1.130}$$

and the hyperbolic trajectory corresponding to Eq. (1.130) is shown in Fig. 1.8. Note that a hyperbola consists of two curves. The first curve is the trajectory of the spacecraft with the focus at point O, where O is the location of the planet. The second curve is a vacant trajectory with a corresponding vacant focus at O'. Furthermore, because $e > 1$, the radius approaches infinity for a value of $\nu = \nu_\infty < \pi$, where ν_∞ is obtained by setting the denominator of Eq. (1.130) to zero. Thus, ν_∞ is obtained by solving

$$1 + e\cos\nu_\infty = 0 \tag{1.131}$$

from which ν_∞ is obtained as

$$\nu_\infty = \cos^{-1}\left(-\frac{1}{e}\right). \tag{1.132}$$

The value of ν_∞ can then be used to determine an angle β that defines the slopes of the two asymptotes of the hyperbolic orbit. The angle β is given as

$$\beta = \pi - \nu_\infty. \tag{1.133}$$

Therefore,

$$\cos\beta = \cos(\pi - \nu_\infty) = \cos\pi\cos\nu_\infty + \sin\pi\sin\nu_\infty = -\cos\nu_\infty = \frac{1}{e} \tag{1.134}$$

from which the angle β is obtained as

$$\beta = \cos^{-1}\left(\frac{1}{e}\right). \tag{1.135}$$

The angles ν_∞ and β are shown in Fig. 1.8. Next, different from an elliptic orbit, the vacant focus O' lies to the right of the focus O (where, again, O is the location of the planet). The fact that the vacant focus O' lies to the right of the focus O (where the planet is located) and $e > 1$ for a hyperbola, the periapsis and apoapsis radii are given from Eqs. (1.46) and (1.47), respectively, as

$$r_p = \frac{p}{1+e} > 0, \tag{1.136}$$

$$r_a = \frac{p}{1-e} < 0. \tag{1.137}$$

Then, because the apoapsis radius is less than zero for a hyperbolic orbit, the semi-major axis for a hyperbolic orbit is

$$a = \frac{p}{1-e^2} < 0. \tag{1.138}$$

It is noted, however that, consistent with Eq. (1.136), the periapsis radius for a hyperbolic orbit is given as

$$r_p = a(1-e) > 0 \tag{1.139}$$

while the semi-latus rectum is

$$p = a(1-e^2) \tag{1.140}$$

where it is noted that both r_p and p are greater than zero because a is less than zero while both $1-e$ and $1-e^2$ are negative. In other words, because $a < 0$ and $e > 1$ for a hyperbolic orbit, the periapsis radius and the semi-latus rectum are both positive. Moreover, from the geometry of the hyperbola it is the case that

$$c^2 = a^2 + b^2 \tag{1.141}$$

where b is defined to be the semi-minor axis of the hyperbola as shown in Fig. 1.8. It is also noted that, because $a < 0$, the energy on a hyperbolic orbit is given as

$$\mathcal{E} = -\frac{\mu}{2a} > 0 \tag{1.142}$$

Finally, the fact that $a < 0$ gives rise to the fact that the distance between the foci O and O' is $-2ae$ while the distance between the periapsis and the apoapsis is $-2a$. As stated, the geometry of the hyperbola is given in Fig. 1.8.

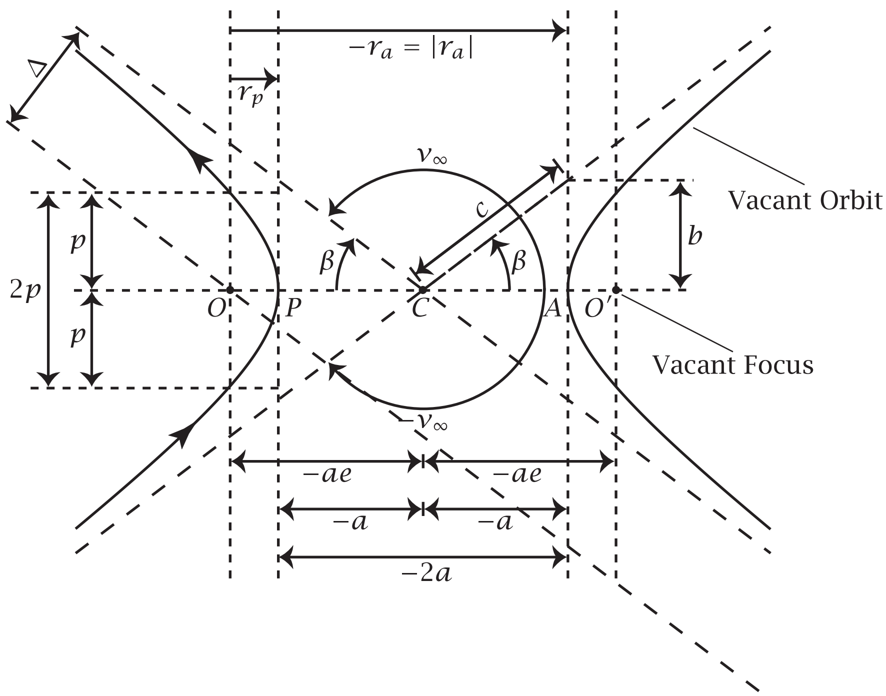


Figure 1.8 Hyperbolic orbit that arises from the solution of the orbit equation given in Eq. (1.45) for the case where $e > 1$.

Problems for Chapter 1

1–1 Consider the following two Earth orbits:

- Orbit 1: Periapsis Radius = $r_{p1} = r_p$; Semi-Major Axis = a_1.
- Orbit 2: Periapsis Radius = $r_{p2} = r_p$; Semi-Major Axis = $a_2 > a_1$.

Using the information provided, determine the following information:

(a) The orbit that has the larger speed at periapsis.

(b) The orbit that has the larger speed at apoapsis.

1–2 A spacecraft is in an Earth orbit whose periapsis altitude is 500 km and whose apoapsis altitude is 800 km. Assuming that the radius of the Earth is $R_e = 6378.145$ km and that the Earth gravitational parameter is $\mu = 398600\ \text{km}^3 \cdot \text{s}^{-2}$, determine the following quantities related to the orbit of the spacecraft:

(a) The semi-major axis.

(b) The eccentricity.

(c) The semi-latus rectum.

(d) The magnitude of the specific angular momentum.

(e) The speed of the spacecraft at periapsis and apoapsis.

1–3 Consider two equatorial orbits, A and B, about a planet. Suppose further that both orbits share the same line of apsides, that the periapses of both orbits are located at the same point (denoted P), that the apoapsis radius of orbit A is smaller than the apoapsis radius of orbit B, and that both orbits have the same semi-minor axes. The goal is to be able to spend as much time as possible from a spacecraft visualizing a point Q fixed to the planet such that the direction OQ lies along the direction from the planet to the apoapsis. Which of the two orbits will enable the spacecraft the longer visualization time of point Q? Justify your answer.

1–4 Spacecraft 1 is in an equatorial circular Earth orbit with a radius r_1. Spacecraft 2 is in a different equatorial Earth orbit from Spacecraft 1 with a periapsis speed v_{p2}. Assuming that the semi-major axes of the orbits of both spacecraft are the same, what is the eccentricity of the orbit of Spacecraft 2 in terms of the information provided.

1–5 A spacecraft is in orbit about the Earth. At a given point on the orbit, the speed, radius, and flight path angle of the spacecraft are $7.5\ \text{km} \cdot \text{s}^{-1}$, 9500 km, and 18 deg, respectively. Determine the following quantities:

(a) The true anomaly.

(b) the eccentricity of the orbit.

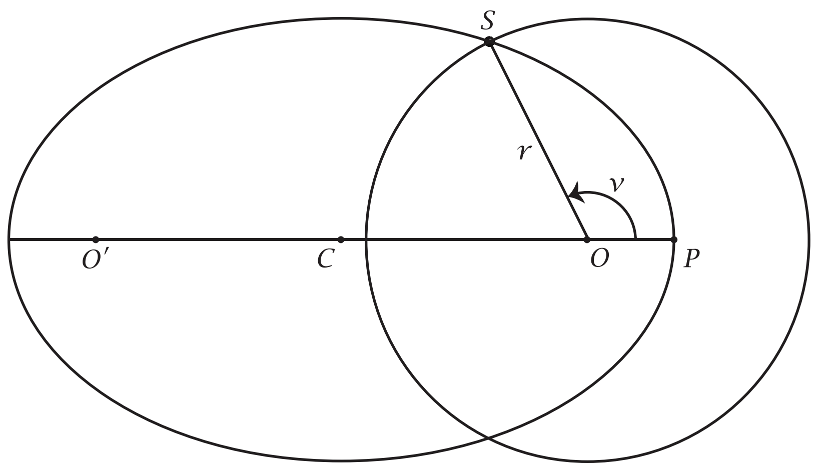
(c) the orbital energy.

1–6 Let $\mathcal{I}$ be a central body (planet) and assume that $\mathcal{I}$ is an inertial reference frame. The position and velocity of a spacecraft relative to the center of $\mathcal{I}$ expressed in planet-centered inertial coordinates at a time t_0 are given in canonical units (that is, units where $\mu = 1$), respectively, as

$$[\mathbf{r}]_{\mathcal{I}} = \frac{1}{20}\begin{bmatrix} -12 \\ -20 \\ 15 \end{bmatrix} \quad , \quad [{}^{\mathcal{I}}\mathbf{v}]_{\mathcal{I}} = \frac{1}{20}\begin{bmatrix} 16 \\ -9 \\ 9 \end{bmatrix}.$$

Determine the inertial acceleration of the spacecraft at t_0 in planet-centered inertial coordinates.

1–7 Consider a spacecraft moving in an elliptic orbit. Determine the value of the true anomaly ν such that the speed on the elliptic orbit at a radius r is equal to the speed on a circular orbit of radius r.



1–8 Consider the definition of the flight path angle, γ, as given in Section 1.4.3. Furthermore, let $\mathbf{r}$ and ${}^{\mathcal{I}}\mathbf{v}$ be position and inertial velocity of a spacecraft relative to a planet, respectively. Expressing all quantities in the basis $\{\mathbf{u}_r, \mathbf{u}_\nu, \mathbf{u}_z\}$ as defined in Eq. (1.75) derive the following:

(a) The result given in Eq. (1.107), that is, derive the expression

$$\tan\gamma = \frac{e\sin\nu}{1 + e\cos\nu}.$$

(b) The maximum and minimum values of the flight path angle on an orbit.

1–9 Let $\mathcal{I}$ be a central body (planet) and assume that $\mathcal{I}$ is an inertial reference frame. The position of a spacecraft relative to the center of the planet and the inertial velocity of the spacecraft expressed in planet-centered inertial coordinates at a time t_0 are given in canonical units (that is, units where the gravitational parameter $\mu = 1$), respectively, as

$$[\mathbf{r}]_{\mathcal{I}}^{\mathsf{T}} = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}, \left[{}^{\mathcal{I}}\mathbf{v}\right]_{\mathcal{I}}^{\mathsf{T}} = \begin{bmatrix} -1/\sqrt{3} & \sqrt{2}/\sqrt{3} & 0 \end{bmatrix}.$$

Determine the following quantities with respect to the orbit of the spacecraft relative to the planet:

(a) The specific angular momentum, ${}^{\mathcal{I}}\mathbf{h}$.

(b) The eccentricity vector. $\mathbf{e}$.

(c) That ${}^{\mathcal{I}}\mathbf{h} \cdot \mathbf{e} = 0$.

(d) The semi-latus rectum, p.

(e) The semi-major axis, a.

(f) The true anomaly at t_0, ν_0.

1–10 Recall from Eqs. (1.55) and (1.56) that the periapsis and apoapsis radii of the orbit of a spacecraft are given as $r_p = a(1-e)$ and $r_a = a(1+e)$, respectively. Therefore, at some point on the orbit between periapsis and apoapsis the radius, r, must be equal to the semi-major axis, a. Assuming a gravitational parameter μ for the planet, determine

(a) The value of the true anomaly when $r = a$.

(b) The speed of the spacecraft at the point when $r = a$.

1–11 A spacecraft is in orbit relative to the Earth, $\mathcal{E}$, and the Earth is considered an inertial reference frame. The specific mechanical energy of the spacecraft is $\mathcal{C} = -2 \times 10^8 \mathrm{ft}^2 \cdot \mathrm{s}^{-2}$ and the orbital eccentricity is $e = 0.2$. Determine

(a) The magnitude of the specific angular momentum, h.

(b) The semi-latus rectum, p.

(c) The semi-major axis, a.

(d) The periapsis radius, r_p.

(e) The apoapsis radius, r_a.

1–12 An Earth-orbiting weather satellite has an orbital eccentricity $e = 0.1$ and a periapsis altitude of 370 km. Determine the following quantities related to the orbit of the spacecraft:

(a) The apoapsis altitude (that is, $h_a = r_a - R_e$ where $R_e = 6378.145$ km is the radius of the Earth).

(b) The specific mechanical energy, $\mathcal{C}$.

(c) The magnitude of the specific angular momentum, h.

(d) The semi-latus rectum.

1–13 An unidentified Earth-orbiting space object is found to have an altitude of 4000 km and is moving with an inertial speed 800 $\mathrm{m \cdot s^{-1}}$ at a flight path angle $\gamma = 0$. Determine

(a) The specific mechanical energy, $\mathcal{C}$.

(b) The magnitude of the specific angular momentum, h.

(c) The semi-latus rectum, p.

(d) The periapsis radius, r_p.

(e) The apoapsis radius, r_a.

1–14 Let $\mathcal{I}$ be a central body (planet) and assume that $\mathcal{I}$ is an inertial reference frame. The position and velocity of a spacecraft relative to the center of $\mathcal{I}$ expressed in planet-centered inertial coordinates at a time t_0 are given in canonical units (that is, units where $\mu = 1$), respectively, as

$$[\mathbf{r}]_{\mathcal{I}}^{\mathsf{T}} = \begin{bmatrix} -0.6 & -1 & 0.75 \end{bmatrix}, \left[{}^{\mathcal{I}}\mathbf{v}\right]_{\mathcal{I}}^{\mathsf{T}} = \begin{bmatrix} 0.8 & -0.45 & 0.45 \end{bmatrix}.$$

Determine the following quantities with respect to the orbit of the spacecraft relative to the planet:

(a) The specific angular momentum, ${}^{\mathcal{I}}\mathbf{h}$.

(b) The eccentricity vector. $\mathbf{e}$.

(c) That ${}^{\mathcal{I}}\mathbf{h} \cdot \mathbf{e} = 0$.

(d) The semi-major axis, a.

(e) The semi-latus rectum, p.

(f) The true anomaly at t_0, ν_0.

1–15 An extra-terrestrial object is found to be approaching Earth when its geocentric radius is 403000 km, its true anomaly is 151 deg, and its Earth-relative speed is $2.25\ \mathrm{km} \cdot \mathrm{s}^{-1}$. Assuming that the Earth is approximated as an inertial reference frame and that the gravitational parameter of the Earth is $\mu = 398600\ \mathrm{km}^3 \cdot \mathrm{s}^{-2}$, determine the following quantities related to the orbit of the object relative to the Earth:

(a) The eccentricity.

(b) The periapsis altitude.

(c) The periapsis speed.

1–16 Please answer true or false to each of the following statements related to the properties of the two-body differential equation derived in class (assuming for simplicity that the orbit is elliptic):

(a) The specific angular momentum is orthogonal to the eccentricity vector.

(b) The specific angular momentum lies in the orbit plane.

(c) The eccentricity vector lies in the orbit plane.

(d) The position of the spacecraft relative to the planet, $\mathbf{r}$, is largest in magnitude when $\mathbf{r}$ lies along the eccentricity vector.

(e) The specific mechanical energy of the orbit is $\mathcal{E} = \dfrac{\mu(1-e^2)}{2p}$.

Chapter 2

The Orbit in Space

2.1 Introduction

The focus of Chapter 1 was to derive the two-body differential equation and to describe the key properties of the motion of a spacecraft in the orbital plane. In particular, the orbit equation derived in Chapter 1 provides the solution of the two-body differential equation from the perspective of the orbit plane. Using the solution of the two-body differential equation led to the property that the orbit is a conic section, with the form of the conic section being either a circle, ellipse, parabola, or hyperbola) depending upon the value of the eccentricity of the orbit. While the information in the orbit plane is essential, it provides only a two-dimensional representation of the motion of a spacecraft. Note, however, that the spacecraft moves in three-dimensional Euclidean space. Therefore, describing motion in the orbital plane is insufficient because the orbital plane itself has an orientation in three-dimensions. Observing that a conic section is a collection of more than three non-collinear points that lie a constant distance from one another, the orbit is a rigid body and, thus, could have any orientation in three-dimensional Euclidean space. Consequently, in order to determine the location of the spacecraft in three-dimensional Euclidean space, it is necessary to know the size and shape of the conic section, the orientation of the conic section in $\mathbb{E}^3$ relative to a reference orientation, and the location of the spacecraft on the conic section.

This chapter describes those parameters required to provide a complete description of the orbit in three dimensions. In particular, the focus of this chapter will be on the development of the *classical orbital elements* that define the size, shape, and orientation of the orbit along with the location of the spacecraft on the orbit itself. The classical orbital elements consist of six quantities called the *semi-major axis*, the *eccentricity*, the *longitude of the ascending node*, the *inclination*, the *argument of the periapsis*, and the *true anomaly*. These semi-major axis and eccentricity define the size and shape of the orbit and were already defined in Chapter 1. The longitude of the ascending node, the inclination, and the argument of the periapsis define a 3-1-3 Euler angle sequence that describes the orientation of the orbit in three-dimensional Euclidean space. Finally, the true anomaly, which was also defined in Chapter 1, describes the location on the orbit relative to the eccenticity vector (which, as described in Chapter 1, is a vector that is fixed in the orbital plane and lies along the direction from the focus of the orbit to the periapsis). Using the definitions developed in this chapter, transformations are developed that enable computation of the classical orbital elements from the position and inertial velocity of the spacecraft when the position and inertial velocity are expressed in planet-centered inertial (PCI) coordinates. Conversely, transformations are developed that enable computation of the position and inertial velocity in planet-centered inertial coordinates given the orbital elements. Finally, a related goal is to describe a method that can be used in computer software for performing the aforementioned computations.

2.2 Coordinate Systems

The first step in describing the orbit of a spacecraft in $\mathbb{E}^3$ is to define relevant coordinate systems from which measurements can be taken in order to provide a quantitative description of the parameters used to describe the orbit. In the context of the Earth, which will be the basis for a quantitative description of the orbit of a spacecraft, the following three coordinate systems, each of which is fixed in an inertial reference frame $\mathcal{I}$, are used to quantify the orbit of a spacecraft: (1) heliocentric-ecliptic coordinates; (2) Earth-centered inertial (ECI) coordinates; and (3) perifocal coordinates. A detailed description of each of these coordinate systems is now provided. It is noted for completeness that the methods developed in this section can be used with any central body that can be approximated as an inertial reference frame, but for convenience the Earth is used as the basis of the analysis.

2.2.1 Heliocentric-Ecliptic Coordinates

The first coordinate system used to quantify the orbit of a spacecraft in $\mathbb{E}^3$ is the *heliocentric-ecliptic coordinate system*. The heliocentric-ecliptic coordinate system is shown in Fig. 2.1 has its origin at the center of the Sun. Now, the Earth moves in a near-circular orbit relative to the Sun. Moreover, the orbit swept out by the Earth relative to the Sun forms what is known as the *ecliptic plane*. The Earth, however, is tipped at an angle of approximately 23.44 deg relative to the ecliptic plane and, thus, the equatorial plane of the Earth is also rotated by 23.44 deg relative to the ecliptic plane. The equatorial plane of the Earth lies within what is known as the *celestial equator* (that is, the equator of the Earth lies in the same plane as that of the celestial equator). The celestial equator and the equatorial plane of the Earth together with the tilt of the Earth relative to the ecliptic plane is shown in Fig. 2.2.

Suppose now the Sun is approximated as an inertial reference frame and the center of the Sun is the point from which all distances are measured. Although using this assumption the motion of the Earth is now observed relative to an inertial reference frame (that inertial reference frame being the Sun), the choice of an inertially fixed coordinate system is arbitrary, the only requirement being that any right-handed orthonormal basis chosen be fixed to the Sun. For convenience, the following inertially fixed basis is chosen. First, let $\mathbf{X}_\epsilon$ be the unit vector that points from the Earth to the Sun on the first day of spring as shown in Fig. 2.1. As seen from the Earth, the Sun will lie directly over the First Point of Aries at the vernal equinox, and the Sun would be said to be entering Aries at this instant of time. In other words, at the instant of time when the Earth is located at the vernal equinox, the direction $\mathbf{X}_\epsilon$ points towards a point known as the *First Point of Aries* (where the First Point of Aries is denoted Υ). The vernal equinox is one of two instants of time where the celestial equator meets the ecliptic plane (the other instant of time being the autumnal equinox).

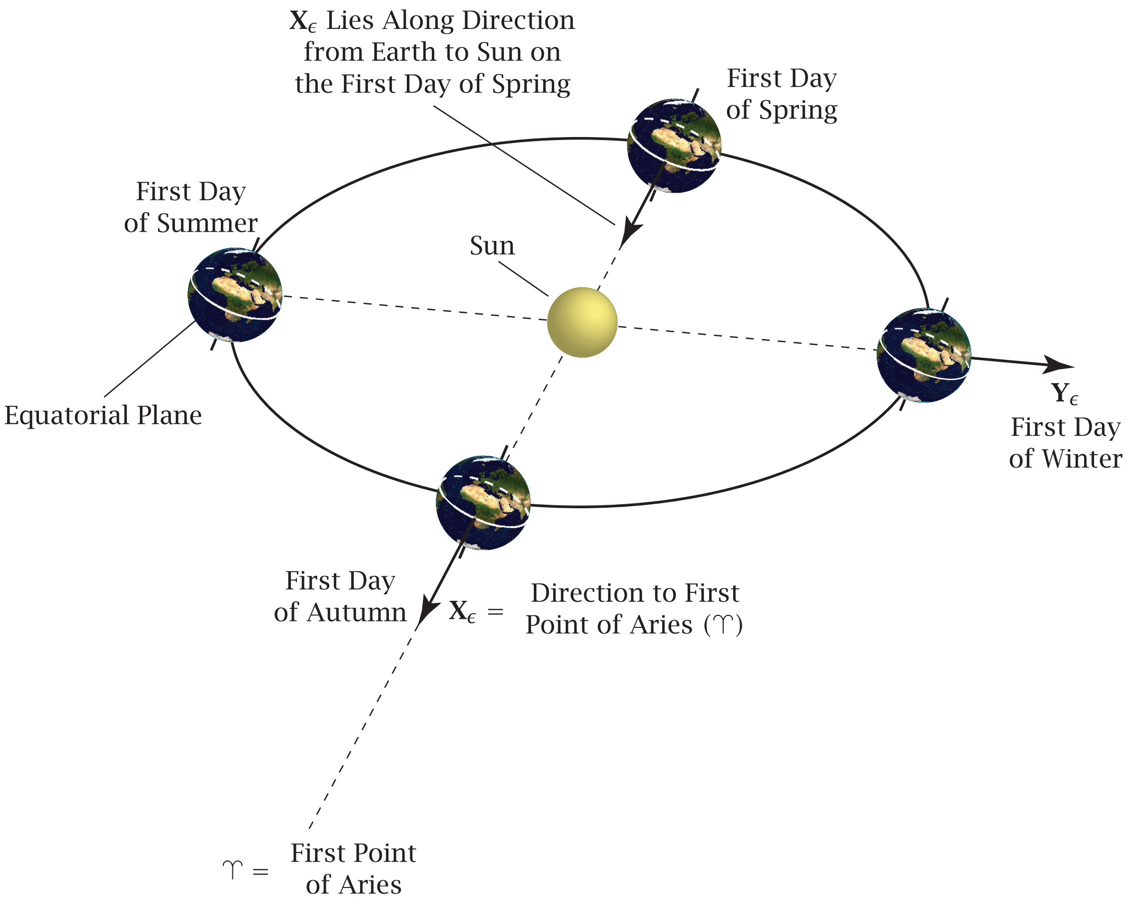


Figure 2.1 Heliocentric-ecliptic coordinate system based on the seasons of the Earth. The unit vector $\mathbf{X}_\epsilon$ is defined as the direction from the Earth to the Sun when the Earth is at the vernal equinox.

2.2.2 Earth-Centered Inertial (ECI) Coordinates

The second coordinate system that is used to quantify the orbit of a spacecraft in $\mathbb{E}^3$ is the *Earth-centered inertial* (ECI) coordinate system. The ECI coordinate system has an origin located at the center of the Earth. The fundamental plane of motion is the Earth equator (which, as stated above, is coincident with the celestial equator) while the ecliptic plane is tilted by an angle of approximately 23.44 deg from the Earth equatorial plane. The first principal direction in the ECI coordinate system is defined as $\mathbf{I}_x$, where $\mathbf{I}_x = \mathbf{X}_\epsilon$ and it is recalled that $\mathbf{X}_\epsilon$ is the unit vector that points from the Earth to the Sun on the first day of spring [which means that $\mathbf{I}_x$ points toward the First Point of Aries (♈)]. Next, the third principal direction is defined as $\mathbf{I}_z$ and lies in the direction from the center of the Earth toward the North pole of the Earth. Finally, the second principal direction, $\mathbf{I}_y$, completes the right-handed system, that is, $\mathbf{I}_y = \mathbf{I}_z \times \mathbf{I}_x$. Now, it is important to note that the ECI coordinate system is actually not fixed in an inertial reference frame because all distances are measured from the center of the Earth and the center of the Earth is not an inertially fixed point. For the purposes of a spacecraft

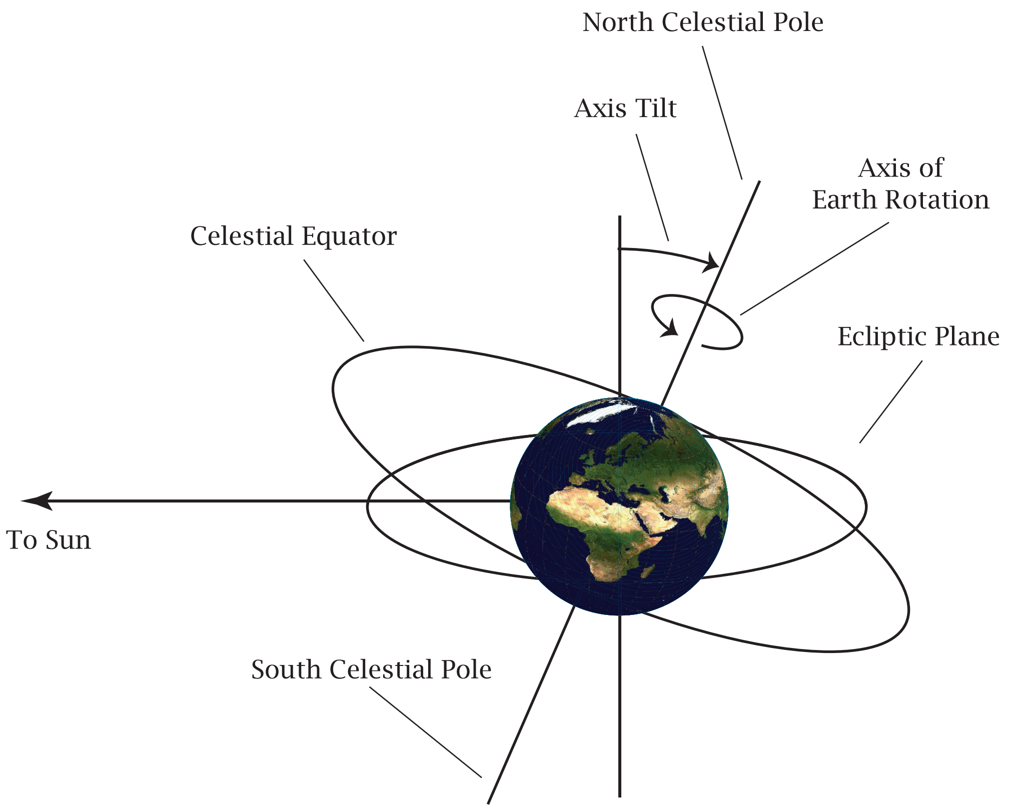


Figure 2.2 Celestial equator and the ecliptic plane of the Earth.

in motion relative to the Earth, it is assumed that the duration over which observations of the motion of a spacecraft is observed is sufficiently small so that the center of the Earth does not move significantly. It is noted, however, that the basis $\left\{\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z\right\}$ is inertially fixed because the direction $\mathbf{I}_x$ is defined at a particular instant of time (in this case, the instant of the vernal equinox) and $\mathbf{I}_z$ is inertially fixed because it is assumed that the rotation of the Earth takes place about an inertially fixed direction. The ECI coordinate system is shown schematically in Fig. 2.3.

2.2.3 Perifocal Coordinates

The third coordinate system that is used in the context of describing the orbit of a spacecraft in $\mathbb{E}^3$ is the *perifocal coordinate system*. Similar to the ECI coordinate system, the perifocal coordinate system has its origin at the center of the Earth. Next, the perifocal basis $\left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$ was already defined in in Eq. (1.74) of Chapter 1, where $\mathbf{p}_x = \mathbf{e}/\|\mathbf{e}\| = \mathbf{e}/e$, $\mathbf{p}_z = {}^{\mathcal{I}}\mathbf{h}/\|{}^{\mathcal{I}}\mathbf{h}\| = {}^{\mathcal{I}}\mathbf{h}/h$, and $\mathbf{p}_y = \mathbf{p}_z \times \mathbf{p}_x$. A two-dimensional projection of the perifocal coordinates into the orbit plane is shown in Fig. 2.4.

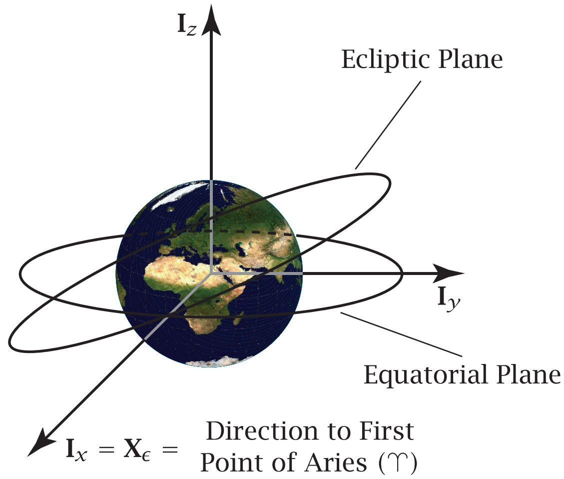


Figure 2.3 Earth-centered inertial (ECI) coordinates. The direction $\mathbf{I}_x$ is the same as the direction $\mathbf{X}$ used to define the heliocentric-ecliptic coordinates shown in Fig. 2.1.

2.3 Orbital Elements

Consider again the Earth-centered inertial (ECI) coordinate system as defined in Section 2.2.2. Suppose further that the inertial reference frame $\mathcal{I}$ is taken to be one such that the original of the Earth together with the basis $\{\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z\}$ as defined Section 2.2.2 are fixed in $\mathcal{I}$. In the study of the two-body problem as given in Chapter 1, it was shown that when the planet was considered to be the inertial reference frame $\mathcal{I}$ that the vectors ${}^{\mathcal{I}}\mathbf{h}$ and $\mathbf{e}$ were shown to be fixed in $\mathcal{I}$. Suppose now that a third vector fixed in $\mathcal{I}$, denoted $\mathbf{n}$, is defined as

$$\mathbf{n} = \mathbf{I}_z \times {}^{\mathcal{I}}\mathbf{h}. \tag{2.1}$$

It can be seen that, because $\mathbf{I}_z$ and ${}^{\mathcal{I}}\mathbf{h}$ are each fixed in $\mathcal{I}$, that $\mathbf{n}$ must also be fixed in $\mathcal{I}$. The vectors ${}^{\mathcal{I}}\mathbf{h}$, $\mathbf{e}$, and $\mathbf{n}$ are now used to define a set of six quantities called the *orbital elements* where the orbital elements are used to parameterize the location of a spacecraft in orbit relative to a planet (in this case that planet is the Earth). The six orbital elements are given as follows:

(i) a, semi-major axis: a constant that defines the size of the orbit;

(ii) e, eccentricity: a constant that defines the shape of the orbit;

(iii) Ω, longitude of the ascending node: a constant that defines the angle between the first principal direction $\mathbf{I}_x$ and the line of nodes;

(iv) i, orbital inclination: a constant that defines the angle between the third principal direction $\mathbf{I}_z$ and specific angular momentum, ${}^{\mathcal{I}}\mathbf{h}$;

(v) ω, argument of the periapsis: the angle from between the line of nodes and the eccentricity vector;

(vi) ν, true anomaly: the angle from the eccentricity vector to the position of the spacecraft relative to the planet.

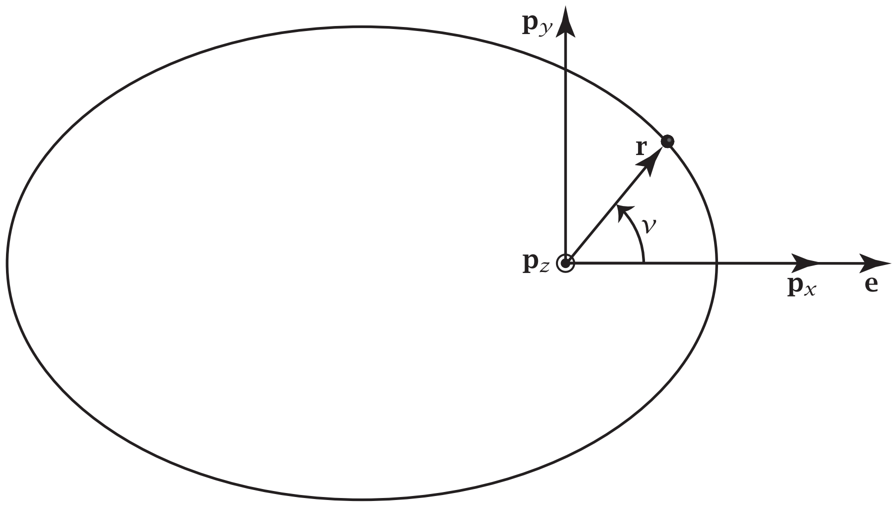


Figure 2.4 Perifocal coordinates shown as a two-dimensional into the orbit plane.

Figure 2.5 provides a schematic of the six orbital elements visualized meausred relative to the ECI coordinate system described in Section 2.2.2. Using the aforementioned description of the orbital elements, in the next section a method is derived for computing the orbital elements given the position and velocity measured in terms of the ECI coordinate system.

2.4 Determining Orbital Elements from Position and Velocity

Using the position and inertial velocity of spacecraft, denoted $\mathbf{r}$ and ${}^{\mathcal{I}}\mathbf{v}$, respectively, the orbital elements are now computed. First, recall from Eq. 1.17 that the specific angular momentum is given as

$$ {}^{\mathcal{I}}\mathbf{h} = \mathbf{r} \times {}^{\mathcal{I}}\mathbf{v}. \tag{2.2} $$

The semi-latus rectum is then obtained as

$$ p = \frac{h^2}{\mu}, \tag{2.3} $$

where $h = \|{}^{\mathcal{I}}\mathbf{h}\|$. Using the semi-latus rectum from Eq. (2.3), the semi-major axis is then obtained as

$$ \boxed{a = \frac{p}{1 - e^2}.} \tag{2.4} $$

Next, the eccentricity vector is given from Eq. (1.33) as

$$ \mathbf{e} = \frac{{}^{\mathcal{I}}\mathbf{v} \times {}^{\mathcal{I}}\mathbf{h}}{\mu} - \frac{\mathbf{r}}{r}, \tag{2.5} $$

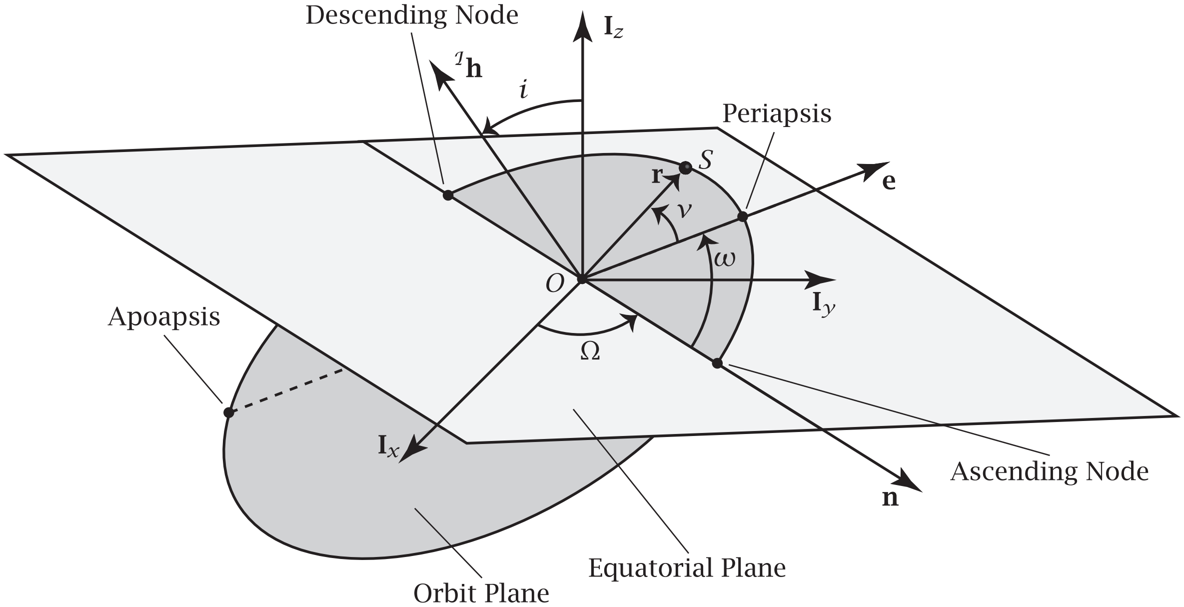


Figure 2.5 Schematic of orbital elements $(a, e, \Omega, i, \omega, \nu)$ relative to the Earth-centered inertial (ECI) coordinate system described in Section 2.2.2.

where $r = \|\mathbf{r}\|$. The eccentricity of the orbit is then given as

$$\boxed{e = \|\mathbf{e}\|.} \tag{2.6}$$

Next, the longitude of the ascending node, Ω, is defined as the angle from $\mathbf{I}_x$ to the line of nodes, where the line of nodes is obtained from Eq. (2.1) as

$$\mathbf{n} = \mathbf{I}_z \times {}^{\mathcal{I}}\mathbf{h}. \tag{2.7}$$

Because $\mathbf{n}$ lies in the $\{\mathbf{I}_x, \mathbf{I}_y\}$–plane, it is seen that the tangent of the longitude of the ascending node is obtained as

$$\tan\Omega = \frac{\mathbf{n}\cdot\mathbf{I}_y}{\mathbf{n}\cdot\mathbf{I}_x} = \frac{\mathbf{n}\cdot\mathbf{I}_y}{\mathbf{n}\cdot\mathbf{I}_x}. \tag{2.8}$$

In order to obtain a value of Ω that is valid for all four quadrants, it is necessary to compute the inverse tangent of Eq. (2.8) using a four-quadrant inverse tangent. In terms of the four-quadrant inverse tangent, the longitude of the ascending node Ω is computed as

$$\boxed{\Omega = \tan^{-1}(\mathbf{n}\cdot\mathbf{I}_y, \mathbf{n}\cdot\mathbf{I}_x).} \tag{2.9}$$

Now, it is noted that the angle Ω obtained in Eq. (2.9) will lie on the interval $[-\pi, \pi]$ (that is $\Omega \in [-\pi, \pi]$). Generally speaking, however, it is desirable for the angle Ω to lie on the interval $[0, 2\pi]$. In order to ensure that $\Omega \in [0, 2\pi]$, is it necessary to check the sign of the angle Ω obtained in Eq. (2.9). If the sign of Ω is *negative*, then 2π is added to the result. This additional check on Ω is performed as follows:

$$\boxed{\text{if } \Omega < 0 \text{ then } \Omega = \Omega + 2\pi.} \tag{2.10}$$

where, it is noted again that the additional check in Eq. (2.10) is included in order ensure that $\Omega \in [0, 2\pi]$.

Next, the inclination, i, is the angle from $\mathbf{I}_z$ to ${}^{\mathcal{I}}\mathbf{h}$. In order to compute the inclination it is necessary to decompose the specific angular momentum ${}^{\mathcal{I}}\mathbf{h}$ into a component along $\mathbf{I}_z$ and a component orthogonal to $\mathbf{I}_z$. The component of ${}^{\mathcal{I}}\mathbf{h}$ lies along a a vector $\mathbf{b}$ (see Fig. 2.6), where $\mathbf{b}$ lie in the $\left\{\mathbf{I}_x, \mathbf{I}_y\right\}$ plane and is orthogonal to the line of nodes, $\mathbf{n}$. Consequently, the vector $\mathbf{b}$ is given as

$$\mathbf{b} = \mathbf{n} \times \mathbf{I}_z. \tag{2.11}$$

Note, however, that because $\mathbf{n}$ is not a unit vector, the $\mathbf{b}$ is not a unit either. Thus, $\mathbf{b}$ must be normalized to be a unit vector as

$$\mathbf{u} = \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\mathbf{n} \times \mathbf{I}_z}{\|\mathbf{n} \times \mathbf{I}_z\|}. \tag{2.12}$$

Now, because $\mathbf{n}$ and $\mathbf{I}_z$ are orthogonal to one another, it follows that

$$\|\mathbf{n} \times \mathbf{I}_z\| = \|\mathbf{n}\| \cdot \|\mathbf{I}_z\| = \|\mathbf{n}\| = n. \tag{2.13}$$

Substituting the result of Eq. (2.13) into (2.12), the unit vector $\mathbf{u}$ is given as

$$\mathbf{u} = \frac{\mathbf{n} \times \mathbf{I}_z}{n}. \tag{2.14}$$

Note that the vectors $\mathbf{I}_z$ and $\mathbf{u}$ form a plane in which the specific angular momentum ${}^{\mathcal{I}}\mathbf{h}$ lies such that i is the angle from $\mathbf{I}_z$ to ${}^{\mathcal{I}}\mathbf{h}$. Using Fig. 2.6 as a guide, it is seen that the tangent of the inclination is given as

$$\tan i = \frac{{}^{\mathcal{I}}\mathbf{h} \cdot \mathbf{u}}{{}^{\mathcal{I}}\mathbf{h} \cdot \mathbf{I}_z} = \frac{{}^{\mathcal{I}}\mathbf{h} \cdot \dfrac{\mathbf{n} \times \mathbf{I}_z}{n}}{{}^{\mathcal{I}}\mathbf{h} \cdot \mathbf{I}_z} \tag{2.15}$$

Then, because n might be zero, the numerator and nominator of Eq. (2.15) are multiplied by n to give

$$\tan i = \frac{{}^{\mathcal{I}}\mathbf{h} \cdot [\mathbf{n} \times \mathbf{I}_z]}{n\left[{}^{\mathcal{I}}\mathbf{h} \cdot \mathbf{I}_z\right]} \tag{2.16}$$

The inclination is then obtained using a four-quadrant inverse tangent as

$$\boxed{i = \tan^{-1}\left({}^{\mathcal{I}}\mathbf{h} \cdot [\mathbf{n} \times \mathbf{I}_z], n[{}^{\mathcal{I}}\mathbf{h} \cdot \mathbf{I}_z]\right).} \tag{2.17}$$

It is noted in Eq. (2.17) that, unlike the case for the longitude of the ascending node, because $i \in [0, \pi]$ it is not necessary to change the signs of the arguments in the four-quadrant inverse tangent function of Eq. (2.17).

Next, the argument of the periapsis, ω, is the angle from $\mathbf{n}$ to $\mathbf{e}$. Note, however, that $\mathbf{n}$ and $\mathbf{e}$ are not orthogonal to one another as shown in Fig. 2.7. In order to compute the argument of the periapsis, it is useful to construct the following right-handed orthonormal basis:

$$\begin{aligned} \mathbf{u}_n &= \frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{\mathbf{n}}{n}, \\ \mathbf{u}_h &= \frac{{}^{\mathcal{I}}\mathbf{h}}{h}, \\ \mathbf{u}_{hn} &= \mathbf{u}_h \times \mathbf{u}_n = \frac{{}^{\mathcal{I}}\mathbf{h}}{h} \times \frac{\mathbf{n}}{n} = \frac{{}^{\mathcal{I}}\mathbf{h} \times \mathbf{n}}{hn} \end{aligned} \tag{2.18}$$

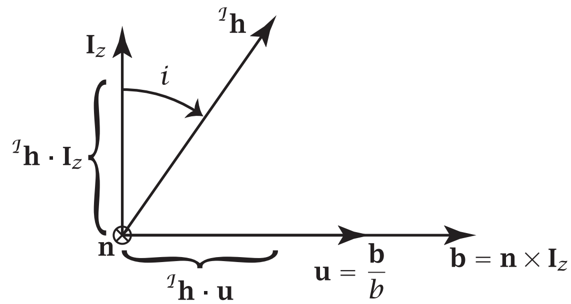


Figure 2.6 Decomposition of specific angular momentum ${}^{\mathcal{I}}\mathbf{h}$ into components along $\mathbf{I}_z$ and orthogonal to $\mathbf{I}_z$ in order to determine the inclination.

where $n = \|\mathbf{n}\|$. Figure 2.7 then shows the components of the eccentricity vector along the directions $\mathbf{u}_h$ and $\mathbf{u}_{hn}$. From Fig. 2.7 it is seen that the tangent of the argument of the periapsis is given as

$$\tan\omega = \frac{\mathbf{e} \cdot \left[\frac{{}^{\mathcal{I}}\mathbf{h} \times \mathbf{n}}{hn}\right]}{\mathbf{e} \cdot \frac{\mathbf{n}}{n}}. \tag{2.19}$$

Then, because either h or n could be zero, the numerator and denominator of Eq. (2.19) are multiplied by hn to give

$$\tan\omega = \frac{\mathbf{e} \cdot [{}^{\mathcal{I}}\mathbf{h} \times \mathbf{n}]}{h[\mathbf{e} \cdot \mathbf{n}]}. \tag{2.20}$$

The argument of the periapsis is then obtained from the four-quadrant inverse tangent as

$$\boxed{\omega = \tan^{-1}\left(\mathbf{e} \cdot [{}^{\mathcal{I}}\mathbf{h} \times \mathbf{n}], h[\mathbf{e} \cdot \mathbf{n}]\right).} \tag{2.21}$$

Now, as was the case for the longitude of the ascending node, the angle ω obtained in Eq. (2.21) will lie on the interval $[-\pi, \pi]$ (that is $\omega \in [-\pi, \pi]$). Generally speaking, however, it is desirable for the angle ω to lie on the interval $[0, 2\pi]$. In order to ensure that $\omega \in [0, 2\pi]$, is it necessary to check the sign of the angle obtained in Eq. (2.21). If the sign of ω is *negative*, then 2π is added to the result. This additional check on ω is performed as follows:

$$\boxed{\text{if } \omega < 0 \text{ then } \omega = \omega + 2\pi.} \tag{2.22}$$

where, it is noted again that the additional check in Eq. (2.22) is included in order ensure that $\omega \in [0, 2\pi]$.

Finally, the true anomaly, ν, is the angle from $\mathbf{e}$ to $\mathbf{r}$. Note, however, that $\mathbf{e}$ and $\mathbf{r}$ are not orthogonal to one another as shown in Fig. 2.8. Now, it is noted that the true anomaly can be computed most conveniently using the previously defined perifocal basis $\left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$ as defined in Eq. (1.74) on page 17. The perifocal basis is restated

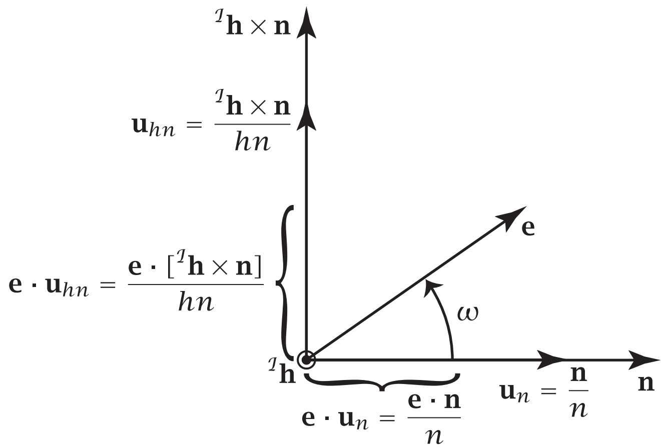


Figure 2.7 Decomposition of the eccentricity vector $\mathbf{e}$ into components along the directions of $\mathbf{n}$ and ${}^{\mathcal{I}}\mathbf{h} \times \mathbf{n}$ in order to determine the argument of the periapsis.

from Eq. (1.74) as

$$\begin{aligned} \mathbf{p}_x &= \frac{\mathbf{e}}{\|\mathbf{e}\|} = \frac{\mathbf{e}}{e}, \\ \mathbf{p}_z &= \frac{{}^{\mathcal{I}}\mathbf{h}}{\|{}^{\mathcal{I}}\mathbf{h}\|} = \frac{{}^{\mathcal{I}}\mathbf{h}}{h}, \\ \mathbf{p}_y &= \mathbf{p}_z \times \mathbf{p}_x. \end{aligned} \tag{2.23}$$

Figure 2.8 shows the components of the spacecraft position $\mathbf{r}$ along the directions $\mathbf{p}_x$ and $\mathbf{p}_y$. Using Fig. 2.8 as a guide, it is seen that the tangent of the true anomaly is given

$$\tan\nu = \frac{\mathbf{r} \cdot \mathbf{p}_y}{\mathbf{r} \cdot \mathbf{p}_x}. \tag{2.24}$$

While, while it seems that the true anomaly could be computed via a four-quadrant inverse tangent (in the same manner that was used to compute Ω, i, and ω), it turns out computing ν requires that Eq. (2.24) be manipulated into a different form. First, because $\mathbf{p}_x$ lies along $\mathbf{e}$ and $\mathbf{p}_z$ lies along ${}^{\mathcal{I}}\mathbf{h}$, it follows that $\mathbf{p}_y$ lies along ${}^{\mathcal{I}}\mathbf{h} \times \mathbf{e}$. Therefore, the perifocal basis can be written as

$$\begin{aligned} \mathbf{p}_x &= \frac{\mathbf{e}}{e}, \\ \mathbf{p}_z &= \frac{{}^{\mathcal{I}}\mathbf{h}}{h}, \\ \mathbf{p}_y &= \mathbf{p}_z \times \mathbf{p}_x = \frac{{}^{\mathcal{I}}\mathbf{h}}{h} \times \frac{\mathbf{e}}{e} = \frac{{}^{\mathcal{I}}\mathbf{h} \times \mathbf{e}}{he}. \end{aligned} \tag{2.25}$$

Consequently, Eq. (2.24) can be written as

$$\tan\nu = \frac{\mathbf{r} \cdot \dfrac{{}^{\mathcal{I}}\mathbf{h} \times \mathbf{e}}{he}}{\mathbf{r} \cdot \dfrac{\mathbf{e}}{e}}. \tag{2.26}$$

Multiplying the numerator and denominator of Eq. (2.26) by he gives

$$\tan \nu = \frac{\mathbf{r} \cdot [{}^{\mathcal{I}}\mathbf{h} \times \mathbf{e}]}{h[\mathbf{r} \cdot \mathbf{e}]}. \tag{2.27}$$

The true anomaly is then obtained from the four-quadrant inverse tangent as

$$\boxed{\nu = \tan^{-1}\left(\mathbf{r} \cdot [{}^{\mathcal{I}}\mathbf{h} \times \mathbf{e}], h[\mathbf{r} \cdot \mathbf{e}]\right),} \tag{2.28}$$

Generally speaking, however, it is desirable for the angle ν to lie on the interval $[0, 2\pi]$. In order to ensure that $\nu \in [0, 2\pi]$, is it necessary to check the sign of the angle obtained in Eq. (2.28). If the sign of ν is *negative*, then 2π is added to the result. This additional check on ν is performed as follows:

$$\boxed{\text{if } \nu < 0 \text{ then } \nu = \nu + 2\pi.} \tag{2.29}$$

where, it is noted again that the additional check in Eq. (2.29) is included in order ensure that $\nu \in [0, 2\pi]$.

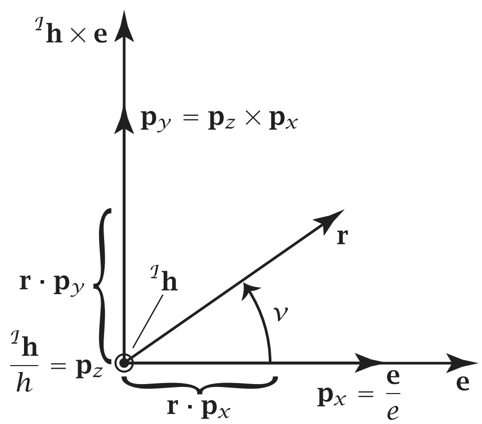


Figure 2.8 Decomposition of the spacecraft position vector $\mathbf{r}$ into components along the directions of $\mathbf{e}$ and ${}^{\mathcal{I}}\mathbf{h} \times \mathbf{e}$ in order to determine the true anomaly.

2.5 Determining Position and Velocity from Orbital Elements

In Section 2.4 a method was developed for determining orbital elements of a spacecraft in moton relative to a planet given the position and the inertial velocity of the spacecraft. In this section the inverse problem is considered, namely, given the orbital elements relative to a planet the objective is to develop a method that determines the position and inertial velocity of the spacecraft. Consider again the ECI coordinate system as defined in Section 2.2.2. Suppose now that the orbital elements $(a, e, \Omega, i, \omega, nu)$ relative to the Earth are known for the spacecraft under consideration and that it is desired to determine the position $\mathbf{r}$ and the inertial velocity ${}^{\mathcal{I}}\mathbf{v}$ expressed in the ECI coordinate system defined by the basis $\mathcal{E} = \left\{\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z\right\}$. The method developed in this section is based on first expressing $\mathbf{r}$ and ${}^{\mathcal{I}}\mathbf{v}$ in the perifocal basis $\mathcal{P} = \left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$ and then transforming the components of these two vectors to the ECI basis $\mathcal{E} = \left\{\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z\right\}$. In order to develop this method a transformation from the perifocal basis $\mathcal{P}$ to the ECI basis $\mathcal{I}$ must be developed.

Let $\mathbf{T}_{\mathcal{P}}^{\mathcal{I}} \in \mathbb{R}^{3\times 3}$ be the matrix that transforms components of a vector expressed in the perifocal basis $\mathcal{P} = \left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$ to the ECI basis $\mathcal{I} = \left\{\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z\right\}$. Now it is known that the perifocal basis $\mathcal{P}$ is fixed in the orbit and that this basis has been obtained through the three transformations by the angles Ω, i, and ω. Moreover, it is known that these three transformations defined by the angles (Ω, i, ω) form a $3-1-3$ Euler angle sequence where the product of the three transformation matrices lead to the overall transformation from the perifocal basis $\mathbf{P}$ to the ECI basis $\mathcal{I}$. The intermediate transformations that correspond to the $3-1-3$ Euler angle sequence defined by (Ω, i, ω) are given as follows:

(i) Transformation 1: “3”-axis transformation by the angle Ω;

(ii) Transformation 2: “1”-axis (resulting from Transformation 1) by the angle i;

(iii) Transformation 3: “3”-axis (resulting from Transformation 2) by the angle ω;

Thus, the components of any vector expressed in the perifocal basis is transformed a new set of components in the ECI basis via the product of the three aforementioned transformations.

2.5.1 Transformation 1 About “3”-Axis via Angle Ω

$$\begin{aligned} \mathbf{n}_x &= \cos\Omega\,\mathbf{I}_x + \sin\Omega\,\mathbf{I}_y, \\ \mathbf{n}_y &= -\sin\Omega\,\mathbf{I}_x + \cos\Omega\,\mathbf{I}_y, \\ \mathbf{n}_z &= \mathbf{I}_z. \end{aligned} \tag{2.30}$$

A schematic of the relationship between the bases $\mathbf{N}$ and $\mathbf{I}$ is given in Fig. 2.9. It is seen

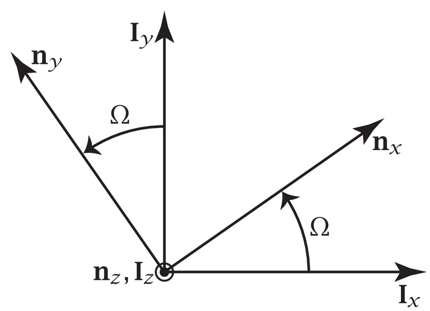


Figure 2.9 Transformation 1 about “3” axis via the longitude of the ascending node, Ω, relating the basis $\mathcal{N} = \left\{\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z\right\}$ to the basis $\mathcal{I} = \left\{\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z\right\}$.

from that Eq. (2.30) that the three basis vectors $\mathbf{n}_x$, $\mathbf{n}_y$, and $\mathbf{n}_z$ expressed in the basis $\mathcal{I}$ are given as

$$[\mathbf{n}_x]_{\mathcal{I}} = \begin{bmatrix} \cos\Omega \\ \sin\Omega \\ 0 \end{bmatrix}, \quad \left[\mathbf{n}_y\right]_{\mathcal{I}} = \begin{bmatrix} -\sin\Omega \\ \cos\Omega \\ 0 \end{bmatrix}, \quad [\mathbf{n}_z]_{\mathcal{I}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{2.31}$$

whereas the three basis vectors $\mathbf{n}_x$, $\mathbf{n}_y$, and $\mathbf{n}_z$ expressed in the basis $\mathcal{N}$ are given as

$$[\mathbf{n}_x]_{\mathcal{N}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathbf{n}_y]_{\mathcal{N}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{n}_z]_{\mathcal{N}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{2.32}$$

Using the result of Eqs. (2.31) and (2.32), the matrix that transforms components of a vector from the basis $\mathcal{N}$ to the basis $\mathcal{I}$ is given as

$$\mathbf{T}_{\mathcal{N}}^{\mathcal{I}} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.33}$$

In other words, given a vector $\mathbf{b}$ expressed in the basis $\mathcal{N}$, denoted $[\mathbf{b}]_{\mathcal{N}}$, that same vector $\mathbf{b}$ expressed in the basis $\mathcal{I}$, denoted $[\mathbf{b}]_{\mathcal{I}}$ is related to $[\mathbf{b}]_{\mathcal{N}}$ as

$$[\mathbf{b}]_{\mathcal{I}} = \mathbf{T}_{\mathcal{N}}^{\mathcal{I}} \, [\mathbf{b}]_{\mathcal{N}} \, . \tag{2.34}$$

2.5.2 Transformation 2: "1"-Axis via Angle i

Let $\mathcal{Q} = \left\{\mathbf{q}_x, \mathbf{q}_y, \mathbf{q}_z\right\}$ be a right-handed orthonormal basis that is related to the basis $\mathcal{N} = \left\{\mathbf{n}_X, \mathbf{n}_y, \mathbf{n}_z\right\}$ as follows:

$$\begin{aligned} \mathbf{q}_x &= \mathbf{n}_x, \\ \mathbf{q}_y &= \cos i \, \mathbf{n}_y + \sin i \, \mathbf{n}_z, \\ \mathbf{q}_z &= -\sin i \, \mathbf{n}_y + \cos i \, \mathbf{n}_z, \end{aligned} \tag{2.35}$$

A schematic of the relationship between the bases **Q** and **N** is given in Fig. 2.10. It is

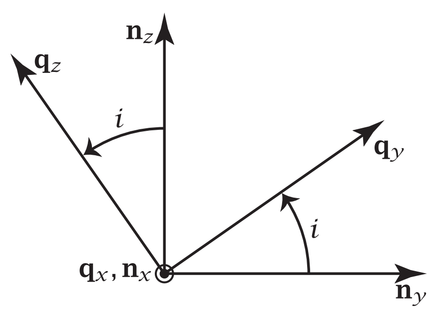


Figure 2.10 Transformation 2 about "1" axis (that results from first transformation) via the inclination, i, relating the basis $\left\{\mathbf{q}_x, \mathbf{q}_y, \mathbf{q}_z\right\}$ to the basis $\left\{\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z\right\}$.

seen from that Eq. (2.35) that the three basis vectors $\mathbf{q}_x$, $\mathbf{q}_y$, and $\mathbf{q}_z$ expressed in the basis $\mathcal{N}$ are given as

$$[\mathbf{q}_x]_{\mathcal{N}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathbf{q}_y]_{\mathcal{N}} = \begin{bmatrix} 0 \\ \cos i \\ \sin i \end{bmatrix}, \quad [\mathbf{q}_z]_{\mathcal{N}} = \begin{bmatrix} 0 \\ -\sin i \\ \cos i \end{bmatrix}, \tag{2.36}$$

whereas the three basis vectors $\mathbf{q}_x$, $\mathbf{q}_y$, and $\mathbf{q}_z$ expressed in the basis $\mathcal{Q}$ are given as

$$[\mathbf{q}_x]_{\mathcal{Q}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \left[\mathbf{q}_y\right]_{\mathcal{Q}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{q}_z]_{\mathcal{Q}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{2.37}$$

Using the result of Eqs. (2.36) and (2.37), the matrix that transforms components of a vector from the basis $\mathcal{Q}$ to the basis $\mathcal{N}$ is given as

$$\mathbf{T}_{\mathcal{Q}}^{\mathcal{N}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix}. \tag{2.38}$$

In other words, given a vector $\mathbf{b}$ expressed in the basis $\mathcal{Q}$, denoted $[\mathbf{b}]_{\mathcal{Q}}$, that same vector $\mathbf{b}$ expressed in the basis $\mathcal{N}$, denoted $[\mathbf{b}]_{\mathcal{N}}$ is related to $[\mathbf{b}]_{\mathcal{Q}}$ as

$$[\mathbf{b}]_{\mathcal{N}} = \mathbf{T}_{\mathcal{Q}}^{\mathcal{N}} [\mathbf{b}]_{\mathcal{Q}}. \tag{2.39}$$

2.5.3 Transformation 3: "3"-Axis via Angle ω

Let $\mathcal{P} = \left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$ be a right-handed orthonormal basis that is related to the basis $\mathcal{Q} = \left\{\mathbf{q}_X, \mathbf{q}_y, \mathbf{q}_z\right\}$ as follows:

$$\begin{aligned} \mathbf{p}_x &= \cos\omega\,\mathbf{q}_x + \sin\omega\,\mathbf{q}_y, \\ \mathbf{p}_y &= -\sin\omega\,\mathbf{q}_x + \cos\omega\,\mathbf{q}_y, \\ \mathbf{p}_z &= \mathbf{q}_z. \end{aligned} \tag{2.40}$$

A schematic of the relationship between the bases **P** and **Q** is given in Fig. 2.11. It is

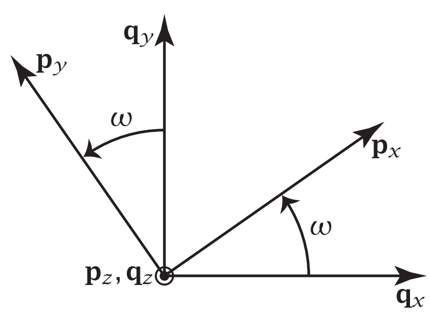


Figure 2.11 Transformation 3 about "3" axis (that results from second transformation) via the inclination, ω, relating the basis $\left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$ to the basis $\left\{\mathbf{q}_x, \mathbf{q}_y, \mathbf{q}_z\right\}$.

seen from that Eq. (2.40) that the three basis vectors $\mathbf{p}_x$, $\mathbf{p}_y$, and $\mathbf{p}_z$ expressed in the basis $\mathcal{Q}$ are given as

$$[\mathbf{p}_x]_{\mathcal{Q}} = \begin{bmatrix} \cos\omega \\ \sin\omega \\ 0 \end{bmatrix}, \quad \left[\mathbf{p}_y\right]_{\mathcal{Q}} = \begin{bmatrix} -\sin\omega \\ \cos\omega \\ 0 \end{bmatrix}, \quad [\mathbf{p}_z]_{\mathcal{Q}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{2.41}$$

whereas the three basis vectors $\mathbf{p}_x$, $\mathbf{p}_y$, and $\mathbf{p}_z$ expressed in the basis $\mathcal{P}$ are given as

$$[\mathbf{p}_x]_{\mathcal{P}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \left[\mathbf{p}_y\right]_{\mathcal{P}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{p}_z]_{\mathcal{P}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{2.42}$$

Using the result of Eqs. (2.41) and (2.42), the matrix that transforms components of a vector from the basis $\mathcal{P}$ to the basis $\mathcal{Q}$ is given as

$$\mathbf{T}_{\mathcal{P}}^{\mathcal{Q}} = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.43}$$

In other words, given a vector $\mathbf{b}$ expressed in the basis $\mathcal{P}$, denoted $[\mathbf{b}]_{\mathcal{P}}$, that same vector $\mathbf{b}$ expressed in the basis $\mathcal{Q}$, denoted $[\mathbf{b}]_{\mathcal{Q}}$ is related to $[\mathbf{b}]_{\mathcal{P}}$ as

$$[\mathbf{b}]_{\mathcal{Q}} = \mathbf{T}_{\mathcal{P}}^{\mathcal{Q}}\,[\mathbf{b}]_{\mathcal{P}}\,. \tag{2.44}$$

2.5.4 Perifocal to Earth-Centered Inertial Transformation

The three transformations described in Sections 2.5.1–2.5.3 are now used to determine the transformation of the components of a vector expressed in the perifocal basis **P** to components of that same vector expressed in the ECI basis $\mathcal{I}$. The transformation of the vector components from **P** to **I** is given by taking the product of the three transformations given in Eqs. (2.44)–2.34. Substituting the result of Eq. (2.44) into (2.39) and substituting that result into Eq. (2.34) gives

$$[\mathbf{b}]_{\mathcal{I}} = \mathbf{T}_{\mathcal{N}}^{\mathcal{I}}\mathbf{T}_{\mathcal{Q}}^{\mathcal{N}}\mathbf{T}_{\mathcal{P}}^{\mathcal{Q}}\,[\mathbf{b}]_{\mathcal{P}}\,, \tag{2.45}$$

where the matrices $\mathbf{T}_{\mathcal{N}}^{\mathcal{I}}$, $\mathbf{T}_{\mathcal{Q}}^{\mathcal{N}}$, and $\mathbf{T}_{\mathcal{P}}^{\mathcal{Q}}$ are defined in Eqs. (2.33), (2.38), and (2.43), respectively. Therefore, the matrix that transforms components of a vector expressed in the perifocal basis to components of the vector expressed in the ECI basis is given as

$$\mathbf{T}_{\mathcal{P}}^{\mathcal{I}} = \mathbf{T}_{\mathcal{N}}^{\mathcal{I}}\mathbf{T}_{\mathcal{Q}}^{\mathcal{N}}\mathbf{T}_{\mathcal{P}}^{\mathcal{Q}}, \tag{2.46}$$

and will be used subsequently to compute the position and inertial velocity of the spacecraft in the ECI basis.

2.5.5 Position and Inertial Velocity in Perifocal Basis

The method for determining position and velocity in the ECI basis is to first express both vectors in the perifocal basis and then to transform the components of these vectors in the perifocal basis to the ECI basis. Recall from Eqs. (1.77) and (1.78) on page 18 that the spacecraft position and inertial velocity can be written, respectively, as

$$\mathbf{r} = r\mathbf{u}_r, \tag{2.47}$$

$$^{\mathcal{I}}\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\nu}\mathbf{u}_\nu. \tag{2.48}$$

where $\mathcal{U} = \{\mathbf{u}_r, \mathbf{u}_\nu, \mathbf{u}_z\}$ is a basis that is rotated by the angle ν relative to the perifocal basis $\mathcal{P} = \left\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right\}$. Therefore, the basis $\mathcal{U}$ is related to the basis $\mathcal{P}$ as

$$\begin{aligned} \mathbf{u}_r &= \cos\nu\mathbf{p}_x + \sin\nu\mathbf{p}_y, \\ \mathbf{u}_\nu &= -\sin\nu\mathbf{p}_x + \cos\nu\mathbf{p}_y, \end{aligned} \tag{2.49}$$

Equation (2.49) can then be used to derive expressions for $\mathbf{r}$ and ${}^{\mathcal{I}}\mathbf{v}$. First, using Eq. (2.47), the position of the spacecraft can be expressed in the perifocal basis as

$$\mathbf{r} = r\cos\nu\mathbf{p}_x + r\sin\nu\mathbf{p}_y. \tag{2.50}$$

which implies that

$$\boxed{[\mathbf{r}]_{\mathcal{P}} = \begin{bmatrix} r\cos\nu \\ r\sin\nu \\ 0 \end{bmatrix}.} \tag{2.51}$$

Next, the expressions for $\dot{\nu}$ and $\dot{r}$ from Eqs. (1.83) and (1.84), respectively, on pages 18 and 19 can be substituted into Eq. (2.48) to obtain

$$\begin{aligned} {}^{\mathcal{I}}\mathbf{v} &= \frac{pe\sin\nu}{(1+e\cos\nu)^2}\dot{\nu}\mathbf{u}_r + \frac{p}{1+e\cos\nu}\dot{\nu}\mathbf{u}_\nu \\ &= \frac{p^2e\sin\nu}{p(1+e\cos\nu)^2}\frac{h}{r^2}\mathbf{u}_r + r\frac{h}{r^2}\mathbf{u}_\nu \\ &= \frac{r^2e\sin\nu}{p}\frac{h}{r^2}\mathbf{u}_r + \frac{h}{r}\mathbf{u}_\nu \\ &= \frac{eh\sin\nu}{p}\mathbf{u}_r + \frac{h}{r}\mathbf{u}_\nu \\ &= \frac{e\sqrt{\mu p}\sin\nu}{p}\mathbf{u}_r + \frac{h(1+e\cos\nu)}{p}\mathbf{u}_\nu \\ &= \frac{e\sqrt{\mu p}\sin\nu}{p}\mathbf{u}_r + \frac{\sqrt{\mu p}(1+e\cos\nu)}{p}\mathbf{u}_\nu \\ &= \sqrt{\frac{\mu}{p}}[e\sin\nu\mathbf{u}_r + (1+e\cos\nu)\mathbf{u}_\nu]. \end{aligned} \tag{2.52}$$

where the identity $p = h^2/\mu$ from Eq. (2.3) on page 38 has been used in Eq. (2.52). Then, substituting Eq. (2.49) into (2.52) gives

$$\begin{aligned} {}^{\mathcal{I}}\mathbf{v} &= \sqrt{\frac{\mu}{p}}[e\sin\nu(\cos\nu\mathbf{p}_x + \sin\nu\mathbf{p}_y) + (1+e\cos\nu)(-\sin\nu\mathbf{p}_x + \cos\nu\mathbf{p}_y)] \\ &= \sqrt{\frac{\mu}{p}}[e\sin\nu\cos\nu\mathbf{p}_x + e\sin^2\nu\mathbf{p}_y - \sin\nu\mathbf{p}_x - e\cos\nu\sin\nu\mathbf{p}_x \\ &\quad + \cos\nu\mathbf{p}_y + e\cos^2\nu\mathbf{p}_y)] \\ &= \sqrt{\frac{\mu}{p}}[-\sin\nu\mathbf{p}_x + (e+\cos\nu)\mathbf{p}_y]. \end{aligned} \tag{2.53}$$

Therefore, the inertial velocity expressed in the perifocal basis $\mathcal{P}$ is given as

$$\boxed{\left[{}^{\mathcal{I}}\mathbf{v}\right]_{\mathcal{P}} = \sqrt{\frac{\mu}{p}}\begin{bmatrix} -\sin\nu \\ e+\cos\nu \\ 0 \end{bmatrix}} \tag{2.54}$$

2.5.6 Position and Inertial Velocity in Earth-Centered Inertial Basis

The results of Sections 2.5.1–2.5.5 can now be used to determine the position and inertial velocity of the spacecraft expressed in the Earth-centered inertial (ECI) basis. First, using the perifocal to ECI transformation given in Eq. (2.46) together with the spacecraft position expressed in the perifocal basis as given in Eq. (2.51), the position and inertial velocity of the spacecraft in the ECI basis are given, repectively, as

$$[\mathbf{r}]_{\mathcal{I}} = \mathbf{T}_{\mathcal{P}}^{\mathcal{I}}[\mathbf{r}]_{\mathcal{P}}, \tag{2.55}$$

$$\left[{}^{\mathcal{I}}\mathbf{v}\right]_{\mathcal{I}} = \mathbf{T}_{\mathcal{P}}^{\mathcal{I}}\left[{}^{\mathcal{I}}\mathbf{v}\right]_{\mathcal{P}}. \tag{2.56}$$

$$\tag{2.57}$$

It is important to note in Eqs. (2.55) and (2.56) that the transformation matrix $\mathbf{T}_{\mathcal{P}}^{\mathcal{I}}$ is a composite transformation arising from the product of the matrices $\mathbf{T}_{\mathcal{N}}^{\mathcal{I}}$, $\mathbf{T}_{\mathcal{Q}}^{\mathcal{N}}$, and $\mathbf{T}_{\mathcal{P}}^{\mathcal{Q}}$ as given in Eq. (2.46). Finally, it is noted again that the individual transformation matrices $\mathbf{T}_{\mathcal{N}}^{\mathcal{I}}$, $\mathbf{T}_{\mathcal{Q}}^{\mathcal{N}}$, and $\mathbf{T}_{\mathcal{P}}^{\mathcal{Q}}$ are functions of the longitude of the ascending node, Ω, the orbital inclination, i, and the argument of the periapsis, ω.

Problems for Chapter 2

2–1 Using the results derived in Section 2.4, develop a MATLAB function that takes the spacecraft position and inertial velocity expressed in planet-centered inertial (PCI) coordinates, denoted $\mathcal{I}$, along with the planet gravitational parameter as inputs and produces the six orbital elements as outputs. The inputs to the code should be two column vectors, one that contains the position, $[\mathbf{r}]_{\mathcal{I}}$, and another that contains the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}]_{\mathcal{I}}$, and a scalar that contains the gravitational parameter, μ. The inputs $[\mathbf{r}]_{\mathcal{I}}$ and $[{}^{\mathcal{I}}\mathbf{v}]_{\mathcal{I}}$ should have the form

$$[\mathbf{r}]_{\mathcal{I}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad , \quad [{}^{\mathcal{I}}\mathbf{v}]_{\mathcal{I}} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}.$$

The output of the code should be a six-dimensional column vector that contains the orbital elements in the order of the output orbital elements should be the same as the order given in this chapter, namely, the output should be a column vector of the form

$$\boldsymbol{\Theta} = \begin{bmatrix} a \\ e \\ \Omega \\ i \\ \omega \\ \nu \end{bmatrix}.$$

The MATLAB function should be set up so that it could be provided to an independent user of the code and produce the required outputs given the required inputs in the format stated.

2–2 Using the results derived in Section 2.5, develop a MATLAB function that takes as inputs the six orbital elements and the planet gravitational parameter and produces as outputs the position and inertial velocity of the spacecraft expressed in a planet-centered inertial (PCI) coordinate system. The input orbital elements should be a column vector of the form

$$\boldsymbol{\Theta} = \begin{bmatrix} a \\ e \\ \Omega \\ i \\ \omega \\ \nu \end{bmatrix}.$$

The outputs of the code should be two column vectors, one that contains the position, $[\mathbf{r}]_{\mathcal{I}}$ and another that contains the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}]_{\mathcal{I}}$. The outputs $[\mathbf{r}]_{\mathcal{I}}$ and $[{}^{\mathcal{I}}\mathbf{v}]_{\mathcal{I}}$ should have the form

$$[\mathbf{r}]_{\mathcal{I}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad , \quad [{}^{\mathcal{I}}\mathbf{v}]_{\mathcal{I}} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}.$$

The MATLAB function should be set up so that it could be provided to an independent user of the code and produce the required outputs given the required inputs in the format stated.

2–3 While the orbital elements are defined for most cases, in certain cases one or more of the orbital elements is not well defined. Revisiting the results of this chapter, determine those cases where each orbital element may not be well defined, and provide a justification as to why that orbital element is not defined for that particular case.

2–4 Show that the rates of change of the specific angular momentum, ${}^{\mathcal{I}}\mathbf{h}$, the eccentricity vector $\mathbf{e}$, and the line of nodes, $\mathbf{n}$ associated with the orbit of a spacecraft relative to a planet $\mathcal{I}$ are zero when the planet is considered an inertial reference frame and all distances are measured relative to the center of the planet.

2–5 The position and inertial velocity of an asteroid relative to the Sun $\mathcal{S}$ with $\mu = 1$ (that is, the gravitational parameter of the planet is unity) at a time t_0 are given are given, respectively, as

$$[\mathbf{r}]_{\mathcal{S}} = \begin{bmatrix} 0.7 \\ 0.6 \\ 0.3 \end{bmatrix} \text{AU} \quad , \quad [{}^{\mathcal{S}}\mathbf{v}]_{\mathcal{S}} = \begin{bmatrix} -0.8 \\ 0.8 \\ 0 \end{bmatrix} \text{AU} \cdot \text{TU}^{-1}.$$

Determine

(a) The six orbital elements for the orbit of the asteroid.

(b) The orbital period.

(c) The semi-latus rectum.

(d) The specific angular momentum and the magnitude of the specific angular momentum.

(e) The specific mechanical energy.

Is the asteroid potentially hazardous to Earth, that is, could the asteroid impact the Earth?

2–6 Using the orbital elements obtained in Question 2–5, determine the position and inertial velocity of the spacecraft in Sun-centered inertial (SCI) coordinates. **Hint:** the result should be the data that was provided in Question 2–5.

2–7 Consider a spacecraft in orbit relative to the Earth $\mathcal{E}$ where the Earth is considered an inertial reference frame. Let the orbital elements of the orbit of the spacecraft be given as

$$\begin{aligned} a &= 15307.548 \text{ km}, \\ e &= 0.7, \\ \Omega &= 194 \text{ deg}, \\ i &= 39 \text{ deg}, \\ \omega &= 85 \text{ deg}, \\ \nu &= 48 \text{ deg}, \end{aligned}$$

where $\mu = 398600 \text{ km}^3 \cdot \text{s}^{-2}$. Determine the position and inertial velocity of the spacecraft expressed in Earth-centered inertial (ECI) coordinates.

2–8 Consider a spacecraft in orbit relative to the Earth $\mathcal{E}$ where the Earth is considered an inertial reference frame. Let the orbital elements of the orbit of the spacecraft be given as

$$\begin{array}{lcl} a & = & 19133.333 \text{ km}, \\ e & = & 0.5, \\ \Omega & = & 30 \text{ deg}, \\ i & = & 45 \text{ deg}, \\ \omega & = & 45 \text{ deg}, \\ \nu & = & 0 \text{ deg}, \end{array}$$

where $\mu = 398600 \text{ km}^3 \cdot \text{s}^{-2}$. Determine the position and inertial velocity of the spacecraft expressed in Earth-centered inertial (ECI) coordinates.

2–9 Consider a spacecraft in orbit relative to the Earth $\mathcal{E}$ where the Earth is considered an inertial reference frame. Suppose that the semi-major axis and the true anomaly of the spacecraft orbit are given, respectively, as

$$\begin{array}{lcl} a & = & 20000 \text{ km}, \\ e & = & 0.45, \\ \Omega & = & 59 \text{ deg}, \\ i & = & 27 \text{ deg}, \\ \omega & = & 94 \text{ deg}, \\ \nu & = & 58 \text{ deg}, \end{array}$$

where $\mu = 398600 \text{ km}^3 \cdot \text{s}^{-2}$. Determine the position and inertial velocity of the spacecraft expressed in Earth-centered inertial (ECI) coordinates.

2–10 Consider a spacecraft in orbit relative to a planet $\mathcal{I}$ where the planet is considered an inertial reference frame. Let the orbital elements of the orbit of the spacecraft be given in canonical units (that is, $\mu = 1$)

$$\begin{array}{lcl} a & = & 1.6, \\ e & = & 0.4, \\ \Omega & = & 287 \text{ deg}, \\ i & = & 46 \text{ deg}, \\ \omega & = & 28 \text{ deg}, \\ \nu & = & 139 \text{ deg}, \end{array}$$

Determine the position and inertial velocity of the spacecraft expressed in planet-centered inertial (PCI) coordinates.

Chapter 3

The Orbit as a Function of Time

3.1 Introduction

Chapters 1 and 2 developed the properties of the orbit as a function of the location of the spacecraft on the orbit, where the location was defined by a quantity called the true anomaly, where the true anomaly is the angle between the eccentricity vector (that lies along the direction from the focus of the orbit to the periapsis of the orbit) and the direction of the spacecraft position. The description of the orbit in space was divided into two parts. Chapter 1 focused on the two-dimensional representation of location of the spacecraft on an orbit (that is, Chapter 1 focused on the properties of the location of the spacecraft in the orbital plane), while Chapter 2 focused on a three-dimensional representation of the orbit, where the three dimensional parameterization employed the classical orbital elements. While the development in Chapters 1 and 2 provided a representation of the motion of the spacecraft in terms of the location of the spacecraft, excluded from Chapters 1 and 2 was the location of the orbit as a function of time.

The objective of this chapter is provide an approach for determining the location of a spacecraft on an orbit as a functio of time. The approach developed in this chapter relies on an angle called the *eccentric anomaly*, where the eccentric anomaly is defined as the angle from the center of an elliptic orbit to a point on a circle that circumscribes the ellipse and is tangent to the ellipse at the periapis and the apoapsis of the orbit. The definition of the eccentric anomaly is then used to derive a relationship between the eccentric anomaly and the true anomaly. This relationship provides the ability to determine the true anomaly at any point on the orbit if the eccentric anomaly can be determined at that same point on the orbit. Then, using Kepler's second law (which states that equal areas on an orbit are swept out in equal time), a relationship, known as *Kepler's equation*, is derived between the eccentric anomaly and time elapsed on the orbit since the instant that the spacecraft was located at the periapsis of the orbit. A more general version of Kepler's equation is then derived for cases where the spacecraft may have crossed periapsis one or more times en route from an initial point on an orbit to a terminal point on the orbit. It is then discussed that, for the case where it is desired to determine the location of a spacecraft at a later time on an orbit given the amount of time that has elapsed from an initial time, Kepler's equation cannot be solved analytically for the eccentric anomaly and, thus, the eccentric anomaly must be determined iteratively. An iterative procedure that employs Newton's method is then described for determining the eccentric anomaly which, in turn, enables determining the true anomaly. Because the first five orbital elements are constant (as described in Chapter 2), once the true anomaly at a later point in time is known the orbital elements can be tranformed to obtain the position and inertial velocity at that later time on the orbit.

3.2 Eccentric Anomaly as a Function of True Anomaly

The first step in being able to determine the location on an orbit given an elapsed time from a reference time is to define a new quantity called the *eccentric anomaly* denoted E. In order to define the eccentric anomaly, first consider a circle of radius a (where a is the semi-major axis of the orbit) with a center C that circumscribes the orbit and is tangent to the orbit at periapsis and apoapsis as shown in Fig. 3.1. Next, as shown in Fig. 3.1, let A be the point on the aforementioned circle such that the line that passes through A and S (where S is the location of the spacecraft) intersects the major axis at point D (where the line segment AD lies orthogonal to the line segment CD). The eccentric anomaly is then defined as the angle from the direction of periapsis (that is, the direction along the eccentricity vector $\mathbf{e}$ to the direction of CA. Figure 3.1 shows the eccentric anomaly along with the true anomaly.

Using the definition of the eccentric anomaly, E, the next objective is to determine a relationship between E and the true anomaly, ν. Consider again Fig. 3.1 and let x and y be the distances from C to D and D to S, respectively. It is seen from Fig. 3.1 that

$$x = a\cos E, \tag{3.1}$$
$$y = b\sin E, \tag{3.2}$$

where b is the semi-minor axis and is the radius of a circle whose center lies at C and is tangent to the orbit at the points where E takes on the values $\pi/2$ and $3\pi/2$). Solving Eqs. (3.1) and (3.2) for $\cos E$ and $\sin E$ gives

$$\cos E = \frac{x}{a}, \tag{3.3}$$
$$\sin E = \frac{y}{b}. \tag{3.4}$$

Suppose now that the distances from C to D, C to O, and O to D are denoted, respectively, as $\overline{CD}$, $\overline{CO}$, and $\overline{OD}$. First, it is seen from Fig. 3.1 that $\overline{CO} = ae$. Second, $\overline{OD} = a\cos E$. Therefore, $\overline{OD} = \overline{CD} - \overline{CO}$, that is,

$$\overline{OD} = \overline{CD} - \overline{CO} = a\cos E - ae = a(\cos E - e). \tag{3.5}$$

Next, the distance from D to S, denoted $\overline{DS}$, is given as

$$\overline{DS} = b\sin E. \tag{3.6}$$

Then, noting that the the line segments OD and DS are orthogonal to one another, the distance r from O to S, denoted $\overline{OS}$, is obtained from the Pythagorean theorem as

$$r^2 = \overline{OS}^2 = \overline{OD}^2 + \overline{DS}^2 = a^2(\cos E - e)^2 + b^2\sin^2 E. \tag{3.7}$$

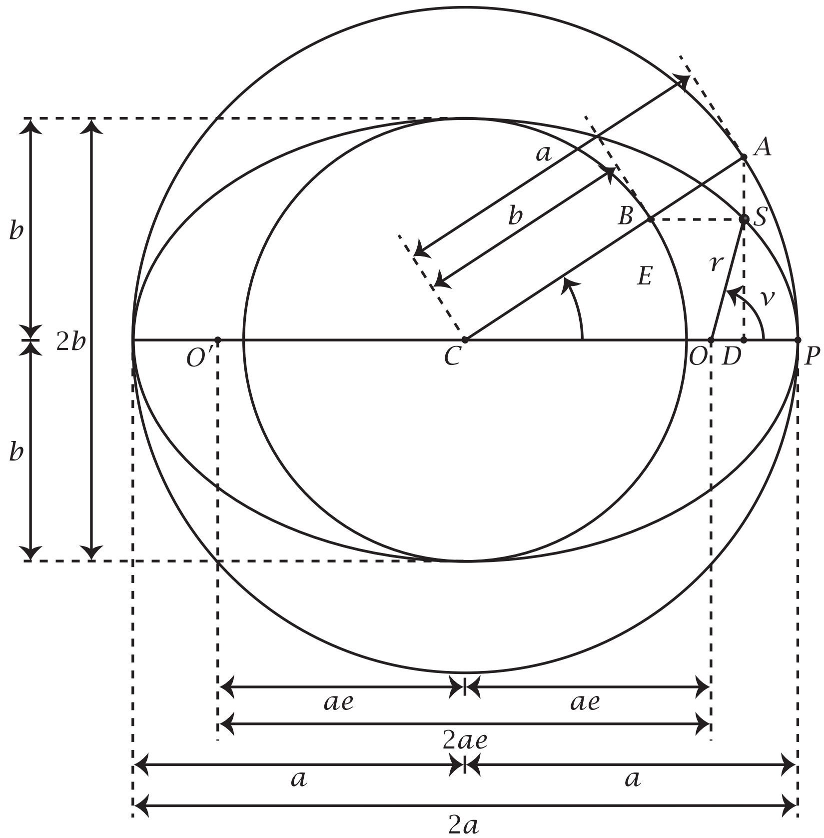


Figure 3.1 Schematic showing the relationship between the eccentric anomaly, E, and the true anomaly, ν, for an elliptic orbit.

Note, however, that the eccentricity of the orbit is given in terms of the semi-major and semi-minor axes as

$$e = \sqrt{1 - \frac{b^2}{a^2}} \tag{3.8}$$

which implies that

$$b^2 = a^2(1 - e^2). \tag{3.9}$$

Substituting the result of Eq. (3.9) into (3.7) gives

$$\begin{aligned} r^2 &= a^2(\cos E - e)^2 + a^2(1 - e^2)\sin^2 E \\ &= a^2\left[\cos^2 E - 2e\cos E + e^2 + (1 - e^2)\sin^2 E\right] \\ &= a^2\left[\cos^2 E - 2e\cos E + e^2 + (1 - e^2)(1 - \cos^2 E)\right] \\ &= a^2\left[\cos^2 E - 2e\cos E + e^2 + (1 - e^2) - (1 - e^2)\cos^2 E\right] \\ &= a^2\left[1 - 2e\cos E + e^2\cos^2 E\right] \\ &= a^2(1 - e\cos E)^2 \end{aligned} \tag{3.10}$$

which implies that

$$\boxed{r = a(1 - e\cos E).} \tag{3.11}$$

Next, it is seen from Fig. 3.1 that the distance x is given as

$$x = ae + r\cos\nu = a\cos E \tag{3.12}$$

which, from Eq. (3.11), gives

$$a\cos E = ae + a(1 - e\cos E)\cos\nu. \tag{3.13}$$

Solving Eq. (3.13) for $\cos\nu$ gives

$$\cos\nu = \frac{\cos E - e}{1 - e\cos E}. \tag{3.14}$$

Then, the sine of the true anomaly can be obtained from the identity

$$\begin{aligned}
\sin^2\nu = 1 - \cos^2\nu = 1 - \left[\frac{\cos E - e}{1 - e\cos E}\right]^2 &= \left[\frac{1 - e\cos E}{1 - e\cos E}\right]^2 - \left[\frac{\cos E - e}{1 - e\cos E}\right]^2 \\
&= \frac{1 - 2e\cos E + e^2\cos^2 E}{(1 - e\cos E)^2} - \frac{\cos^2 E - 2e\cos E + e^2}{(1 - e\cos E)^2} \\
&= \frac{1 - 2e\cos E + e^2\cos^2 E}{(1 - e\cos E)^2} + \frac{-\cos^2 E + 2e\cos E - e^2}{(1 - e\cos E)^2} \\
&= \frac{(1 - e^2) - (1 - e^2)\cos^2 E}{(1 - e\cos E)^2} = \frac{(1 - e^2)(1 - \cos^2 E)}{(1 - e\cos E)^2} = \frac{(1 - e^2)\sin^2 E}{(1 - e\cos E)^2}
\end{aligned} \tag{3.15}$$

which implies that

$$\sin\nu = \frac{\sqrt{1 - e^2}\sin E}{1 - e\cos E}. \tag{3.16}$$

Now, Eqs. (3.14) and (3.16) can be used together to derive a relationship between the eccentric anomaly and the true anomaly that is valid regardless of the quadrant in which any of the angles lie. First, consider the tangent half-angle identity

$$\tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos\theta}{\sin\theta}. \tag{3.17}$$

Applying Eq. (3.17) to the angle ν using the results of Eqs. (3.14) and (3.16) gives

$$\begin{aligned}
\tan\left(\frac{\nu}{2}\right) &= \frac{1 - \dfrac{\cos E - e}{1 - e\cos E}}{\dfrac{\sqrt{1 - e^2}\sin E}{1 - e\cos E}} = \frac{1 - e\cos E - (\cos E - e)}{\sqrt{1 - e^2}\sin E} \\
&= \frac{(1 + e) - (1 + e)\cos E}{\sqrt{1 - e^2}\sin E} = \frac{(1 + e)(1 - \cos E)}{\sqrt{1 - e^2}\sin E} \\
&= \frac{(1 + e)(1 - \cos E)}{\sqrt{(1 + e)(1 - e)}\sin E} = \sqrt{\frac{1 + e}{1 - e}}\frac{1 - \cos E}{\sin E}.
\end{aligned} \tag{3.18}$$

Then, applying Eq. (3.17) again to the angle E, Eq. (3.18) becomes

$$\tan\left(\frac{\nu}{2}\right) = \sqrt{\frac{1 + e}{1 - e}}\tan\left(\frac{E}{2}\right). \tag{3.19}$$

Taking the four-quadrant inverse tangent on both sides of Eq. (3.19) gives

$$\frac{\nu}{2} = \tan^{-1}\left[\sqrt{1+e}\sin\left(\frac{E}{2}\right), \sqrt{1-e}\cos\left(\frac{E}{2}\right)\right] \tag{3.20}$$

Now $E \in [0, 2\pi]$ which implies that $E/2 \in [0, \pi]$. Consequently, $E/2$ lies in either the first or second quadrants which implies that $\nu/2$ in Eq. (3.20) lies on the interval $[0, \pi]$. As a result, $\nu \in [0, 2\pi]$ in Eq. (3.20). The true anomaly on $[0, 2\pi]$ is then obtained simply by multiplying both sides of Eq. (3.20) by two, that is,

$$\boxed{\nu = 2\tan^{-1}\left[\sqrt{1+e}\sin\left(\frac{E}{2}\right), \sqrt{1-e}\cos\left(\frac{E}{2}\right)\right]} \tag{3.21}$$

For completeness, Eq. (3.19) can be rearranged as

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\nu}{2}\right). \tag{3.22}$$

Then, by a similar argument as that used to obtain Eq. (3.21), the relationship for E in terms of ν that results in $E \in [0, 2\pi]$ is given as

$$\boxed{E = 2\tan^{-1}\left[\sqrt{1-e}\sin\left(\frac{\nu}{2}\right), \sqrt{1+e}\cos\left(\frac{\nu}{2}\right)\right]} \tag{3.23}$$

where it is noted that, similar to Eq. (3.21), a four-quadrant inverse tangent is used in Eq. (3.23). Finally, for completeness it is noted that either Eq. (3.21) or (3.23) is valid for $0 \leq e < 1$ regardless of the quadrant in which the angles E and ν lie. Figure 3.2 shows a plot of the true anomaly, ν, as a function of the eccentric anomaly, E, for various values of eccentricity, e, obtained using Eq. (3.21). It can be seen in Fig. 3.2 that Eq. (3.21) produces a true anomaly that lies on $[0, 2\pi]$ given an eccentric anomaly that lies on $[0, 2\pi]$.

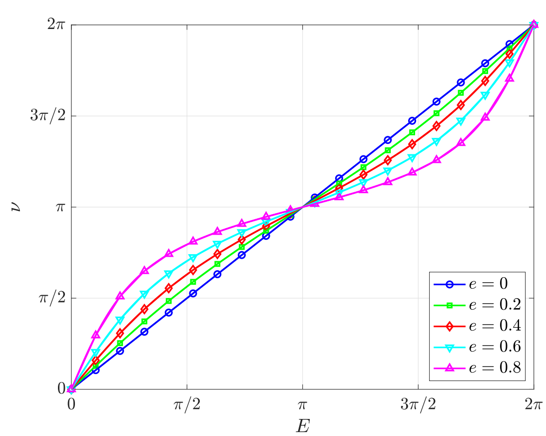


Figure 3.2 True anomaly, ν, as a function of the eccentric anomaly, E, for various values of eccentricity, e, obtained using Eq. (3.21)

3.3 Relating Eccentric Anomaly and Time

Section 3.2 provided a definition of the eccentric anomaly, E, and further determined a relationship between E and the true anomaly, ν, on an orbit. Using the definition of E from Section 3.2, the objective of this section is to determine a relationship between the eccentric anomaly and the elapsed time, $t - t_0$, between an initial and a terminal point on an orbit. The equation that provides the relationship between E and $t - t_0$ is called *Kepler's equation* and provides the basis for determining either the elapsed time given the change in the eccentric anomaly or for determining the change in the eccentric anomaly given an elapsed time between two points on an orbit.

To start the derivation of the relationship between the eccentric anomaly and the time elapsed on an orbit, consider again Eq. (3.11). Then, taking the rate of change of r in Eq. (3.11) gives

$$\dot{r} = ae\dot{E}\sin E. \tag{3.24}$$

Next, taking the rate of change of r from the orbit equation in Eq. (1.45) gives

$$\dot{r} = \frac{pe\dot{\nu}\sin\nu}{(1+e\cos\nu)^2} = \frac{pe\sin\nu}{(1+e\cos\nu)^2}\dot{\nu}. \tag{3.25}$$

Multiplying the numerator and denominator of $\dot{r}$ in Eq. (3.25) gives

$$\dot{r} = \frac{p^2 e\sin\nu}{p(1+e\cos\nu)^2}\dot{\nu} = \frac{p^2}{(1+e\cos\nu)^2}\frac{e\sin\nu}{p}\dot{\nu} = \left(\frac{p}{1+e\cos\nu}\right)^2\frac{e\sin\nu}{p}\dot{\nu}. \tag{3.26}$$

Now, it is noted that

$$r^2 = \left(\frac{p}{1+e\cos\nu}\right)^2. \tag{3.27}$$

Therefore, Eq. (3.26) can then be written as

$$\dot{r} = \frac{r^2 e\sin\nu}{p}\dot{\nu}. \tag{3.28}$$

Next, substituting the expression for $\dot{\nu}$ from Eq. (1.83) into Eq. (3.28) gives

$$\dot{r} = \frac{r^2 e\sin\nu}{p}\frac{h}{r^2} = \frac{eh\sin\nu}{p}. \tag{3.29}$$

Furthermore, observing from Eq. (1.44) that $h = \sqrt{\mu p}$, Eq. (3.29) can be written as

$$\dot{r} = \frac{e\sqrt{\mu p}\sin\nu}{p} = e\sqrt{\frac{\mu}{p}}\sin\nu \tag{3.30}$$

Furthermore, noting from Eq. (1.51) that $p = a(1-e^2)$, Eq. (3.30) becomes

$$\dot{r} = e\sqrt{\frac{\mu}{a(1-e^2)}}\sin\nu \tag{3.31}$$

Equating the results of Eqs. (3.24) and (3.28) leads to the equation

$$ae\dot{E}\sin E = e\sqrt{\frac{\mu}{a(1-e^2)}}\sin\nu \tag{3.32}$$

Now, Eq. (3.2) together with the fact that $y = r\sin\nu$ gives

$$r\sin\nu = b\sin E \tag{3.33}$$

which implies that

$$\sin\nu = \frac{b}{r}\sin E = \frac{a\sqrt{1-e^2}}{r}\sin E. \tag{3.34}$$

Substituting the result of Eq. (3.34) into (3.32) gives

$$ae\dot{E}\sin E = e\sqrt{\frac{\mu}{a(1-e^2)}}\frac{a\sqrt{1-e^2}}{r}\sin E = \frac{ae}{r}\sqrt{\frac{\mu}{a}}\sin E. \tag{3.35}$$

Equation (3.35) can be simplified to obtain

$$r\dot{E} = \sqrt{\frac{\mu}{a}}. \tag{3.36}$$

Then, substituting r from Eq. (3.11) into Eq. (3.36) gives

$$a(1 - e\cos E)\dot{E} = \sqrt{\frac{\mu}{a}}. \tag{3.37}$$

which implies that

$$(1 - e\cos E)\dot{E} = \sqrt{\frac{\mu}{a^3}}. \tag{3.38}$$

Equation (3.38) provides a starting point for deriving a relationship between eccentric anomaly and time.

It is seen that the right-hand side of Eq. (3.38), that is, μ/a^3, is a constant. As a result, Eq. (3.38) and can be integrated as

$$\int (1 - e\cos E)dE = E - e\sin E = \int \sqrt{\frac{\mu}{a^3}}dt = \sqrt{\frac{\mu}{a^3}}t + C \tag{3.39}$$

where C is a constant of integration. The constant C can be evaluated using the fact that $E(t_0) = E_0$ which gives

$$E(t_0) - e\sin E(t_0) = E_0 - e\sin E_0 = \sqrt{\frac{\mu}{a^3}}t_0 + C \tag{3.40}$$

Solving Eq. (3.40) for C gives

$$C = E_0 - e\sin E_0 - \sqrt{\frac{\mu}{a^3}}t_0 \tag{3.41}$$

which implies that

$$E(t) - e\sin E(t) - (E_0 - e\sin E_0) = \sqrt{\frac{\mu}{a^3}}(t - t_0) \tag{3.42}$$

where $E(t)$ is the eccentric anomaly at an arbitrary time t on the orbit. The quantity $E(t) - e\sin E(t)$ is called the *mean anomaly* is denoted $M(t)$, that is,

$$M(t) = E(t) - e\sin E(t). \tag{3.43}$$

Equation (3.42) is called *Kepler's equation*.

While in principal the form of Kepler's equation given in Eq. (3.42) provides a relationship between the time as a function of true anomaly, it is not the most convenient form to use. Instead, Kepler's equation can be re-written in the following way. Suppose that t_P is the time at which the spacecraft was located at periapsis just prior to its location at t_0. Then, at time t_P it must be the case that E is a multiple of 2π, that is, $E(t_P) = 2\pi k,\ (k = 0, \pm 1, \pm 2, \ldots)$. Denoting $E_p = E(t_p)$, the value E_p can arbitrarily be set to zero, that is,

$$E_P = E(t_P) = 0. \tag{3.44}$$

Now, in terms of t_P, $t - t_0$ can be written as

$$t - t_0 = (t - t_P) - (t_0 - t_P). \tag{3.45}$$

Now consider each term in Eq. (3.45) separately. First, consider the term $t_0 - t_P$. Then, applying Eq. (3.42) by replacing t with t_0, replacing t_0 with t_P, and noting that $E(t_0) = E_0$ and $E(t_P) = E_P$ gives

$$E_0 - e\sin E_0 - (E_P - e\sin E_P) = \sqrt{\frac{\mu}{a^3}}(t_0 - t_P). \tag{3.46}$$

But it is seen from Eq. (3.44) that $E_P = 0$ which implies that $\sin E_P = 0$. Therefore, Eq. (3.46) simplifies to

$$E_0 - e\sin E_0 = \sqrt{\frac{\mu}{a^3}}(t_0 - t_P). \tag{3.47}$$

Next, consider the term $t - t_P$. Now, as the spacecraft travels from $E_P = 0$ to $E(t)$, it is possible that it makes one or more complete orbits (revolutions) about the central body. Moreover, it is noted that, every time the spacecraft makes a complete revolution, the eccentric anomaly changes by 2π. Therefore, if the spacecraft makes k revolutions in moving from E_P to $E(t)$, the change in the eccentric anomaly from t_P to t can be written as

$$E(t) = 2\pi k + E, \tag{3.48}$$

where $E \in [0, 2\pi]$ is the eccentric anomaly measured from the periapsis on the incomplete revolution after the spacecraft has crossed periapsis for the k^{th} time. Figure 3.3 provides a schematic of the change in the eccentric anomaly as the spacecraft moves from $E_P = 0$ to $E(t)$ along with more information about the angle E given in Eq. (3.48).

Substituting $E(t)$ in Eq. (3.48) into Eq. (3.42) by replacing t_0 with t_P gives

$$E(t) - e\sin E(t) = 2\pi k + E - e\sin E = \sqrt{\frac{\mu}{a^3}}(t - t_P). \tag{3.49}$$

Then, Eqs. (3.49) and (3.46) can be subtracted to obtain

$$2\pi k + E - e\sin E - (E_0 - e\sin E_0) = \sqrt{\frac{\mu}{a^3}}(t - t_P) - \sqrt{\frac{\mu}{a^3}}(t_0 - t_P) = \sqrt{\frac{\mu}{a^3}}(t - t_0). \tag{3.50}$$

Equation (3.50) can be solved for $t - t_0$ to give

$$\boxed{t - t_0 = \sqrt{\frac{a^3}{\mu}}\left[2\pi k + (E - e\sin E) - (E_0 - e\sin E_0)\right].} \tag{3.51}$$

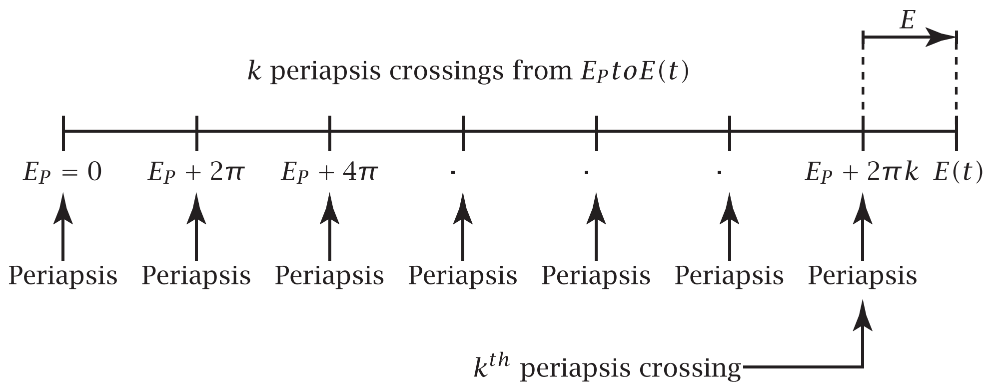


Figure 3.3 Schematic of the change in eccentric anomaly as the spacecraft travels from $E_P = 0$ at time t_P to $E(t)$ at time t and crosses periapsis k times en route from E_P to $E(t)$. The angle E is the true anomaly measured from periapsis after the k^{th} periapsis crossing.

It is noted that Eq. (3.51) is the *general form of Kepler's equation* and provides a relationship between the time elapsed on an orbit and the eccentric anomaly. Effectively, Eq. (3.51) provides a way of solving the two-body differential equation for the location on the orbit at time t given the initial time t_0 and the initial location on the orbit where the initial location on the orbit is given by the initial eccentric anomaly E_0.

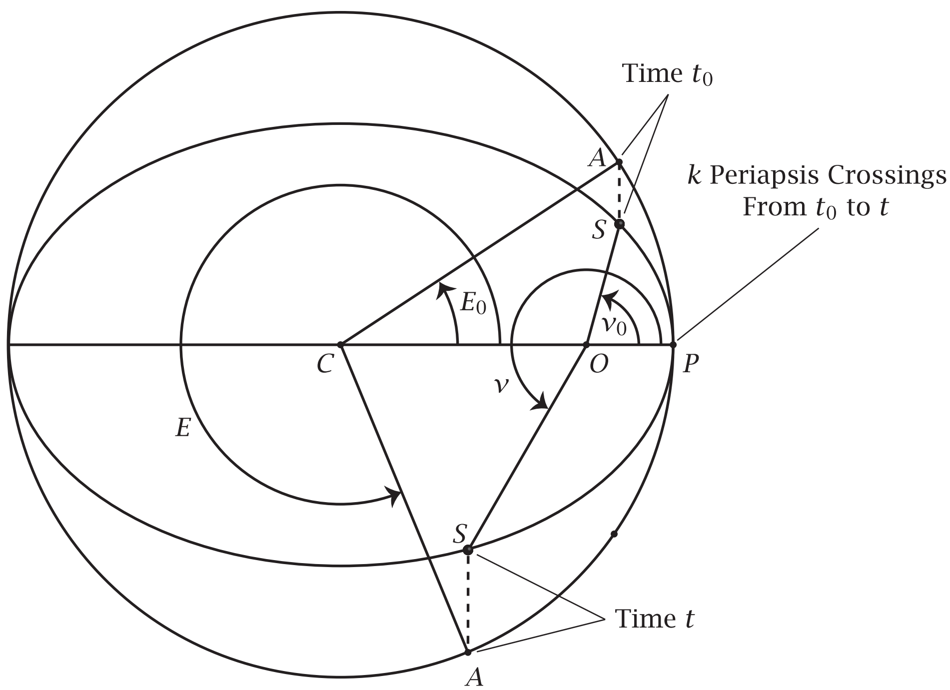


Figure 3.4 Schematic showing two general points on an orbit where the spacecraft is located at point A_0 at time t_0 with a true anomaly and eccentric anomaly ν_0 and E_0, respectively, and is located at point A at time t with a true anomaly and eccentric anomaly ν and E, respectively. The spacecraft crosses periapsis a total of k times en route from point A_0 to point A.

3.4 Solving Kepler's Equation

Equation (3.51) derived in Section 3.3 provides a relationship between the time elapsed as a spacecraft moves en route from an initial location on an orbit at t_0 to a final location on the orbit at time t. Upon further examination, it is seen that Eq. (3.51) can be used in one of two ways. The first possibility is that, given an init Computing $t - t_0$ given the aforementioned information is algebraic in substituting the quantities E_0, E, and k into Eq. (3.51) yields $t - t_0$ directly. The second possibility is that, given the initial time, t_0, the initial eccentric anomaly E_0, and the final time t, Eq. (3.51) can be used to solve for the final eccentric anomaly E. Note, however, that, unlike the situation where the initial and final location information is given as it is desired to obtain $t - t_0$, in this latter situation the final eccentric anomaly E cannot be obtained algebraically because the mean anomaly $M = E - e\sin E$ is a transcendental function of E and, thus, it is not possible to obtain E algebraically using Eq. (3.51).

The final eccentric anomaly, E, given t_0, E_0, and t, can be obtained as follows. Suppose Eq. (3.51) is solved for $E - e\sin E$ as

$$E - e\sin E = \sqrt{\frac{\mu}{a^3}}(t - t_0) - 2\pi k + (E_0 - e\sin E_0). \tag{3.52}$$

It is seen in Eq. (3.52) that all of the quantities on the right-hand side of Eq. (3.52) are known. Thus, the only unknown in Eq. (3.52) is E. Because E cannot be obtained algebraically, it must be obtained numerically using a root-finding method. Suppose the quantity C is defined as

$$C = \sqrt{\frac{\mu}{a^3}}(t - t_0) - 2\pi k + (E_0 - e\sin E_0). \tag{3.53}$$

Then Eq. (3.53) can be written as

$$E = e\sin E + C \tag{3.54}$$

Now, Eq. (3.54) will have a solution provided that the orbit is elliptic, that is, Eq. (3.54) can be solved if $0 \le e < 1$. The solution to Eq. (3.54) is called a *fixed point* and can be obtained using a *fixed-point iteration*. A fixed-point iteration has the general form

$$x^{(k+1)} = f(x^{(k)}), \tag{3.55}$$

where the value x is sought and $x^{(k)}$ is the k^{th} iteration, and $f(x)$ is the function that maps $x^{(k)} \longrightarrow x^{(k+1)}$. In this case f is a function of the eccentric anomaly and is given as

$$f(E) = e\sin E + C. \tag{3.56}$$

It is noted that f is also a function of the paramters a and e, a and e are constants. Applying Eq. (3.55) to (3.54), the fixed-point iteration used to solve Kepler's equation is given as follows:

$$E^{(k+1)} = e\sin E^{(k)} + C. \tag{3.57}$$

Finally, it is noted that two aspects of the fixed-point iteration must be set in order to obtain an accurate solution. First, it is important to choose a good initial guess. As it turns out, the following initial guess works quite well:

$$E^{(0)} = M_0 = E_0 - e\sin E_0, \tag{3.58}$$

where M_0 is the mean anomaly at t_0 [see Eq. (3.43)]. Next, the number of iterations must be specified. It turns out that the proper number of iterations depends upon the value of the eccentricity, e, and is obtained from the following relationship:

$$N = 10\left\lceil \frac{1}{1-e} \right\rceil, \tag{3.59}$$

where $\lceil x \rceil$ is the largest integer in x.

3.5 Method for Determining Location on Orbit

The results of Sections 3.2 and 3.4 can now be combined with the results of Sections 2.4 and 2.5 as described in Chapter 2 to develop a method for determining the position and inertial velocity of a spacecraft expressed in Earth-centered inertial (ECI) coordinates at an arbitrary time t given the position and inertial velocity of the spacecraft expressed in ECI coordinates at an initial time t_0. The following information is required for the method that follows:

- The initial time, t_0.
- The position and inertial velocity of the spacecraft at t_0, $\mathbf{r}(t_0)$ and ${}^{\mathcal{I}}\mathbf{v}(t_0)$.
- The final time t.

Furthermore, in the method that follows let ν_0 be the true anomaly at the initial time t_0. The method for determining $(\mathbf{r}(t), {}^{\mathcal{I}}\mathbf{v}(t))$ then consists of the the following five steps:

1. Given $(\mathbf{r}(t_0), {}^{\mathcal{I}}\mathbf{v}(t_0))$ as described in Section 2.4, compute the orbital elements $(a, e, \Omega, i, \omega, \nu_0)$ of Chapter 2.
2. Solve for E_0 using Eq. (3.21) on page 57 given the initial true anomaly, ν_0 as obtained in Step 1.
3. Solve Eq. (3.54) for E using a fixed-point iteration as described in Section 3.4, where the value C is given in Eq. (3.53).
4. Solve for the true anomaly ν at time t using Eq. (3.23), where E is as obtained in Step 3.
5. Given the orbital elements $(a, e, \Omega, i, \omega, \nu)$, where ν is as obtained in Step 4, compute the final position and inertial velocity, $(\mathbf{r}(t), {}^{\mathcal{I}}\mathbf{v}(t))$, as described in Section 2.4 of Chapter 2.

Problems for Chapter 3

3–1 Let $\mathcal{I}$ be a planet and let O be the center of the planet. Furthermore, assume that $\mathcal{I}$ is an inertial reference frame. The two-body differential equation relative to the center of the planet is given in Eq. (1.15) as

$$^{\mathcal{I}}\mathbf{a} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0},$$

where $\mathbf{r}$ is the position of a spacecraft measured relative to point O and $^{\mathcal{I}}\mathbf{a}$ is the acceleration of the body as viewed by an observer fixed in $\mathcal{I}$. The objective of this question is to write a MATLAB code that integrates the two-body differential equation from an initial time $t = t_0$ to a terminal time $t = t_f$ given that the position, $\mathbf{r}$, and the inertial velocity, $^{\mathcal{I}}\mathbf{v}$, are known at t_0. In order to make it possible to perform the aforementioned integration in MATLAB, consider that $\mathbf{r}$ and $^{\mathcal{I}}\mathbf{v}$ are parameterized in a planet-centered inertial (PCI) coordinate system defined by the center of the planet, that is,

$$\begin{aligned} [\mathbf{r}]_{\mathcal{I}} &= \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \\ [^{\mathcal{I}}\mathbf{v}]_{\mathcal{I}} &= \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}. \end{aligned}$$

Using the PCI parameterization of the position and velocity of the spacecraft, determine the following:

(a) A system of six first-order differential equations for the motion of the spacecraft.

(b) A MATLAB function called twoBodyOde that computes the right-hand side of the six differential equations given in part (a). The MATLAB function should take the following inputs:

- The time, t.
- A 6×1 column matrix $\mathbf{p}$ such that the first three components of $\mathbf{p}$ are $[\mathbf{r}]_{\mathcal{I}}$ and the fourth through sixth components of $\mathbf{p}$ are $[^{\mathcal{I}}\mathbf{v}]_{\mathcal{I}}$.
- The gravitational parameter, μ.

Next, the output of the MATLAB function is a 6×1 column matrix $\dot{\mathbf{p}}$ that contains the right-hand side of the differential equations evaluated at t and $\mathbf{p}$. Note for completeness that the output $\dot{\mathbf{p}}$ must be coded in MATLAB as the variable named "pdot".

(c) Write a MATLAB m-file that, given an initial PCI position and PCI inertial velocity, $[\mathbf{r}]_{\mathcal{I}}$ and $[^{\mathcal{I}}\mathbf{v}]_{\mathcal{I}}$, integrates the MATLAB implementation of the differential equations using the MATLAB differential equation solver ode113. This MATLAB m-file should set up the function ode113 and ode113 should call the function that was written in part (b).

(d) Apply your MATLAB code to the following cases, report your PCI position and inertial velocity at $t = t_f$, and plot the PCI position $[\mathbf{r}]_{\mathcal{I}}$ you obtain with the command plot3 in MATLAB.

(1) $(t_0, t_f) = (1329.16, 3885.73)$ min and

$$[\mathbf{r}(t_0)]_{\mathcal{I}} = \begin{bmatrix} 68524.298 \\ -17345.863 \\ -51486.409 \end{bmatrix} \text{km} \quad , \quad [{}^{\mathcal{I}}\mathbf{v}(t_0)]_{\mathcal{I}} = \begin{bmatrix} -0.578936 \\ 0.957665 \\ 0.357759 \end{bmatrix} \text{km} \cdot \text{s}^{-1}.$$

(2) $(t_0, t_f) = (3.93, 1771.58)$ min and

$$[\mathbf{r}(t_0)]_{\mathcal{I}} = \begin{bmatrix} 2721.965 \\ 3522.863 \\ 5267.244 \end{bmatrix} \text{km} \quad , \quad [{}^{\mathcal{I}}\mathbf{v}(t_0)]_{\mathcal{I}} = \begin{bmatrix} 9.572396 \\ -0.474701 \\ -2.725664 \end{bmatrix} \text{km} \cdot \text{s}^{-1}.$$

(3) $(t_0, t_f) = (242.82, 612.69)$ min and

$$[\mathbf{r}(t_0)]_{\mathcal{I}} = \begin{bmatrix} 6997.56 \\ -34108.00 \\ 20765.49 \end{bmatrix} \text{km} \quad , \quad [{}^{\mathcal{I}}\mathbf{v}(t_0)]_{\mathcal{I}} = \begin{bmatrix} 0.15599 \\ 0.25517 \\ 1.80763 \end{bmatrix} \text{km} \cdot \text{s}^{-1}.$$

(4) $(t_0, t_f) = (616.79, 1880.41)$ min and

$$[\mathbf{r}(t_0)]_{\mathcal{I}} = \begin{bmatrix} 1882.725 \\ 9864.690 \\ 4086.088 \end{bmatrix} \text{km} \quad , \quad [{}^{\mathcal{I}}\mathbf{v}(t_0)]_{\mathcal{I}} = \begin{bmatrix} -5.565367 \\ 5.451548 \\ 2.258105 \end{bmatrix} \text{km} \cdot \text{s}^{-1}.$$

(5) $(t_0, t_f) = (21.02, 1913.38)$ min and

$$\mathbf{r}(t_0) = \begin{bmatrix} -664.699 \\ 8112.75 \\ 4479.81 \end{bmatrix} \text{km} \quad , \quad {}^{\mathcal{I}}\mathbf{v}(t_0) = \begin{bmatrix} -0.87036 \\ -0.068046 \\ -8.290459 \end{bmatrix} \text{km} \cdot \text{s}^{-1}.$$

(6) $(t_0, t_f) = (27, 57)$ min and

$$[\mathbf{r}(t_0)]_{\mathcal{I}} = \begin{bmatrix} -10515.45 \\ -5235.37 \\ 49.1700 \end{bmatrix} \text{km,} \quad , \quad [{}^{\mathcal{I}}\mathbf{v}(t_0)]_{\mathcal{I}} = \begin{bmatrix} -2.10305 \\ -4.18146 \\ 5.56329 \end{bmatrix} \text{km} \cdot \text{s}^{-1}.$$

3–2 Using the results developed in this chapter for the case of an elliptic orbit (that is, $0 \le e < 1$), develop a solver in MATLAB that uses Kepler's equation to determine the position and inertial velocity of a spacecraft in planet-centered inertial (PCI) coordinates at an arbitrary time t, given the position and inertial velocity of the spacecraft in PCI coordinates at an initial time t_0. The inputs to the MATLAB code should be a column vector that contains the initial position, $\mathbf{r}(t_0)$, a second column vector that contains the inertial velocity, ${}^{\mathcal{I}}\mathbf{v}(t_0)$, a scalar that contains the initial time, t_0, a scalar that contains the final time, t, and a scalar that contains the planet gravitational parameter, μ. The outputs to the MATLAB code should be a column vector that contains the position, $\mathbf{r}(t)$ and a second column vector that contains the inertial velocity, ${}^{\mathcal{I}}\mathbf{v}(t)$. The MATLAB function should be set up so that it could be provided to an independent user of the code and produce the required outputs given the required inputs in the format stated.

3–3 Suppose that the position and inertial velocity of a spacecraft in Earth (denoted $\mathcal{I}$) orbit at a time $t_0 = 1329.16$ min are given in Earth-centered inertial (ECI) coordinates as

$$[\mathbf{r}(t_0)]_{\mathcal{I}} = \begin{bmatrix} 68524.298 \\ -17345.863 \\ -51486.409 \end{bmatrix} \text{km} \quad , \quad [{}^{\mathcal{I}}\mathbf{v}(t_0)]_{\mathcal{I}} = \begin{bmatrix} -0.578936 \\ 0.957665 \\ 0.357759 \end{bmatrix} \text{km} \cdot \text{s}^{-1}.$$

Determine the following quantities related to the spacecraft orbit at a time $t = 3885.73$ min:

(a) The eccentric anomaly, E.

(b) The true anomaly, ν.

(c) The ECI position, $[\mathbf{r}(t)]_{\mathcal{I}}$, and the ECI inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(d) Compare the value of $[\mathbf{r}(t)]_{\mathcal{I}}$ and $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ obtained in part (c) to the values obtained using the MATLAB integrator `ode113` (see Question 1).

Finally, perform the following steps to construct a time series of values of PCI position and velocity at various times on the time interval $t \in [t_0, t_f]$:

(e) Divide the time interval $t \in [t_0, t_f]$ into 100 subintervals and store the values in single column matrix (array).

(f) Solve for the ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$, and the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ at each value of time given in the array created in part (e).

(g) Collect the values of ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$ and ECI inertial velocity $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ into an array where each row of each array corresponds to a value of either $[\mathbf{r}(t)]_{\mathcal{I}}$ or $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(h) Display and print the arrays created in part (g).

(i) Plot the array of PCI positions, $[\mathbf{r}]_{\mathcal{I}}$, obtained in part (e) alongside the result of using the MATLAB integrator `ode113` with the MATLAB command `plot3`.

In determining your answers, use $\mu = 398600$ km$^3 \cdot$ s^{-2} as the value for the Earth gravitational parameter.

3–4 Suppose that the position and inertial velocity of a spacecraft in Earth (denoted $\mathcal{I}$) orbit at a time $t_0 = 3.93$ min are given, respectively, as

$$\begin{aligned} \mathbf{r}^{\mathsf{T}}(t_0) &= \begin{bmatrix} 2721.965 & 3522.863 & 5267.244 \end{bmatrix} \text{km}, \\ {}^{\mathcal{I}}\mathbf{v}^{\mathsf{T}}(t_0) &= \begin{bmatrix} 9.572396 & -0.474701 & -2.725664 \end{bmatrix} \text{km} \cdot \text{s}^{-1}. \end{aligned}$$

Determine the following quantities related to the spacecraft orbit at a time $t = 1771.58$ min:

(a) The eccentric anomaly, E.

(b) The true anomaly, ν.

(c) The position, $[\mathbf{r}(t)]_{\mathcal{I}}$, and the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(d) Compare the value of $[\mathbf{r}(t)]_{\mathcal{I}}$ and $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ obtained in part (c) to the values obtained using the MATLAB integrator `ode113` (see Question 1).

Finally, perform the following steps to construct a time series of values of PCI position and velocity at various times on the time interval $t \in [t_0, t_f]$:

(e) Divide the time interval $t \in [t_0, t_f]$ into 100 subintervals and store the values in single column matrix (array).

(f) Solve for the ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$, and the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ at each value of time given in the array created in part (e).

(g) Collect the values of ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$ and ECI inertial velocity $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ into an array where each row of each array corresponds to a value of either $[\mathbf{r}(t)]_{\mathcal{I}}$ or $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(h) Display and print the arrays created in part (g).

(i) Plot the array of PCI positions, $[\mathbf{r}]_{\mathcal{I}}$, obtained in part (e) alongside the result of using the MATLAB integrator `ode113` with the MATLAB command `plot3`.

In determining your answers, use $\mu = 398600\ \mathrm{km}^3 \cdot \mathrm{s}^{-2}$ as the value for the Earth gravitational parameter.

3–5 Suppose that the position and inertial velocity of a spacecraft in Earth (denoted $\mathcal{I}$) orbit at a time $t_0 = 242.82$ min are given, respectively, as

$$
\begin{aligned}
\mathbf{r}^{\mathsf{T}}(t_0) &= \begin{bmatrix} 6997.56 & -34108.00 & 20765.49 \end{bmatrix}\ \mathrm{km}, \\
{}^{\mathcal{I}}\mathbf{v}^{\mathsf{T}}(t_0) &= \begin{bmatrix} 0.15599 & 0.25517 & 1.80763 \end{bmatrix}\ \mathrm{km}\cdot\mathrm{s}^{-1}.
\end{aligned}
$$

Determine the following quantities related to the spacecraft orbit at a time $t = 612.69$ min:

(a) The eccentric anomaly, E.

(b) The true anomaly, ν.

(c) The position, $[\mathbf{r}(t)]_{\mathcal{I}}$, and the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(d) Compare the value of $[\mathbf{r}(t)]_{\mathcal{I}}$ and $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ obtained in part (c) to the values obtained using the MATLAB integrator `ode113` (see Question 1).

Finally, perform the following steps to construct a time series of values of PCI position and velocity at various times on the time interval $t \in [t_0, t_f]$:

(e) Divide the time interval $t \in [t_0, t_f]$ into 100 subintervals and store the values in single column matrix (array).

(f) Solve for the ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$, and the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ at each value of time given in the array created in part (e).

(g) Collect the values of ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$ and ECI inertial velocity $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ into an array where each row of each array corresponds to a value of either $[\mathbf{r}(t)]_{\mathcal{I}}$ or $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(h) Display and print the arrays created in part (g).

(i) Plot the array of PCI positions, $[\mathbf{r}]_{\mathcal{I}}$, obtained in part (e) alongside the result of using the MATLAB integrator `ode113` with the MATLAB command `plot3`.

In determining your answers, use $\mu = 398600\ \mathrm{km}^3 \cdot \mathrm{s}^{-2}$ as the value for the Earth gravitational parameter.

3–6 Suppose that the position and inertial velocity of a spacecraft in Earth (denoted $\mathcal{I}$) orbit at a time $t_0 = 616.79$ min are given, respectively, as

$$\begin{aligned} \mathbf{r}^{\mathsf{T}}(t_0) &= \begin{bmatrix} 1882.725 & 9864.690 & 4086.088 \end{bmatrix} \mathrm{km}, \\ {}^{\mathcal{I}}\mathbf{v}^{\mathsf{T}}(t_0) &= \begin{bmatrix} -5.565367 & 5.451548 & 2.258105 \end{bmatrix} \mathrm{km} \cdot \mathrm{s}^{-1}. \end{aligned}$$

Determine the following quantities related to the spacecraft orbit at a time $t = 1880.41$ min:

(a) The eccentric anomaly, E.

(b) The true anomaly, ν.

(c) The position, $[\mathbf{r}(t)]_{\mathcal{I}}$, and the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(d) Compare the value of $[\mathbf{r}(t)]_{\mathcal{I}}$ and $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ obtained in part (c) to the values obtained using the MATLAB integrator `ode113` (see Question 1).

Finally, perform the following steps to construct a time series of values of PCI position and velocity at various times on the time interval $t \in [t_0, t_f]$:

(e) Divide the time interval $t \in [t_0, t_f]$ into 100 subintervals and store the values in single column matrix (array).

(f) Solve for the ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$, and the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ at each value of time given in the array created in part (e).

(g) Collect the values of ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$ and ECI inertial velocity $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ into an array where each row of each array corresponds to a value of either $[\mathbf{r}(t)]_{\mathcal{I}}$ or $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(h) Display and print the arrays created in part (g).

(i) Plot the array of PCI positions, $[\mathbf{r}]_{\mathcal{I}}$, obtained in part (e) alongside the result of using the MATLAB integrator `ode113` with the MATLAB command `plot3`.

In determining your answers, use $\mu = 398600\ \mathrm{km}^3 \cdot \mathrm{s}^{-2}$ as the value for the Earth gravitational parameter.

3–7 Suppose that the position and inertial velocity of a spacecraft in Earth (denoted $\mathcal{I}$) orbit at a time $t_0 = 21.02$ min are given, respectively, as

$$\begin{aligned} \mathbf{r}^{\mathsf{T}}(t_0) &= \begin{bmatrix} -664.699 & 8112.75 & 4479.81 \end{bmatrix} \mathrm{km}, \\ {}^{\mathcal{I}}\mathbf{v}^{\mathsf{T}}(t_0) &= \begin{bmatrix} -0.87036 & -0.068046 & -8.290459 \end{bmatrix} \mathrm{km} \cdot \mathrm{s}^{-1}. \end{aligned}$$

Determine the following quantities related to the spacecraft orbit at a time $t = 1913.38$ min:

(a) The eccentric anomaly, E.

(b) The true anomaly, ν.

(c) The position, $[\mathbf{r}(t)]_{\mathcal{I}}$, and the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(d) Compare the value of $[\mathbf{r}(t)]_{\mathcal{I}}$ and $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ obtained in part (c) to the values obtained using the MATLAB integrator `ode113` (see Question 1).

Finally, perform the following steps to construct a time series of values of PCI position and velocity at various times on the time interval $t \in [t_0, t_f]$:

(e) Divide the time interval $t \in [t_0, t_f]$ into 100 subintervals and store the values in single column matrix (array).

(f) Solve for the ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$, and the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ at each value of time given in the array created in part (e).

(g) Collect the values of ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$ and ECI inertial velocity $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ into an array where each row of each array corresponds to a value of either $[\mathbf{r}(t)]_{\mathcal{I}}$ or $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(h) Display and print the arrays created in part (g).

(i) Plot the array of PCI positions, $[\mathbf{r}]_{\mathcal{I}}$, obtained in part (e) alongside the result of using the MATLAB integrator `ode113` with the MATLAB command `plot3`.

In determining your answers, use $\mu = 398600\ \text{km}^3 \cdot \text{s}^{-2}$ as the value for the Earth gravitational parameter.

3–8 Suppose that the position and inertial velocity of a spacecraft in Earth (denoted $\mathcal{I}$) orbit at a time $t_0 = 27$ min are given, respectively, as

$$\begin{aligned} \mathbf{r}^{\mathsf{T}}(t_0) &= \begin{bmatrix} -10515.45 & -5235.37 & 49.17 \end{bmatrix} \text{ km}, \\ {}^{\mathcal{I}}\mathbf{v}^{\mathsf{T}}(t_0) &= \begin{bmatrix} -2.10305 & -4.18146 & 5.563290 \end{bmatrix} \text{ km} \cdot \text{s}^{-1}. \end{aligned}$$

Determine the following quantities related to the spacecraft orbit at a time $t = 57$ min:

(a) The eccentric anomaly, E.

(b) The true anomaly, ν.

(c) The position, $[\mathbf{r}(t)]_{\mathcal{I}}$, and the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(d) Compare the value of $[\mathbf{r}(t)]_{\mathcal{I}}$ and $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ obtained in part (c) to the values obtained using the MATLAB integrator `ode113` (see Question 1).

Finally, perform the following steps to construct a time series of values of PCI position and velocity at various times on the time interval $t \in [t_0, t_f]$:

(e) Divide the time interval $t \in [t_0, t_f]$ into 100 subintervals and store the values in single column matrix (array).

(f) Solve for the ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$, and the inertial velocity, $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ at each value of time given in the array created in part (e).

(g) Collect the values of ECI position $[\mathbf{r}(t)]_{\mathcal{I}}$ and ECI inertial velocity $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$ into an array where each row of each array corresponds to a value of either $[\mathbf{r}(t)]_{\mathcal{I}}$ or $[{}^{\mathcal{I}}\mathbf{v}(t)]_{\mathcal{I}}$.

(h) Display and print the arrays created in part (g).

(i) Plot the array of PCI positions, $[\mathbf{r}]_{\mathcal{I}}$, obtained in part (e) alongside the result of using the MATLAB integrator `ode113` with the MATLAB command `plot3`.

In determining your answers, use $\mu = 398600\ \text{km}^3 \cdot \text{s}^{-2}$ as the value for the Earth gravitational parameter.

Chapter 4

Rocket Dynamics

4.1 Introduction

Chapters 1–3 focused on the motion of a spacecraft under the influence of a central body gravitational force. Using these concepts as a starting point, the next objective is to determine way to change the orbit of a spacecraft by applying a propulsive (thrust) force to the spacecraft. The term *orbit transfer* itself refers to utilizing external forces to change the orbit of a spacecraft, the primary force amongt those external forces being thrust. In order to develop the orbit transfers it is first necessary to study the motion of a spacecraft that is subject to a thrust force. The general term for the study of the motion of a vehicle subject to a thrust force is called *rocket dynamics.* In this chapter the physics and mathematics of the differential equation that governs the motion of a rocket subject to a thrust force is derived. Then, this differential equation for the motion of a rocket is solved, leading to the rocket equation that provides relationship between the change in velocity, also known as Δv, as a function of the fuel that is consumed. The rocket equation is first derived for a vehicle whose thrust is finite. Then, the *impulsive-thrust approximation* is developed where it is assumed that the thrus is extremely large and can be approximated by infinite thrust that is capable of changing velocity instantaneously. The impulsive-thrust model developed in this chapter is then used as the basis of impulsive orbit transfer in Chapter 5.

4.2 Rocket Equation

The first step in developing the fundamental impulsive orbit transfers is to describe the dynamics associate with the expenditure of propellant that arises when a vehicle in motion is subject to a thrust (propulsive) force. Specifically, consider a vehicle moving through free space where at an instant of time t the mass and the inertial velocity of the vehicle are given, respectively, by $m(t)$ and ${}^{\mathcal{I}}\mathbf{v}(t)$. Then the linear momentum of the system at time t, denoted ${}^{\mathcal{I}}\mathbf{p}(t)$, is given as

$$ {}^{\mathcal{I}}\mathbf{p}(t) = m(t)\,{}^{\mathcal{I}}\mathbf{v}(t) \tag{4.1} $$

A schematic of the configuration of the vehicle at time t is shown in Fig. 4.1. Next, suppose that the resultant *external* force acting on the vehicle is $\mathbf{F}$. Suppose further that, due to propellent consumption, the mass of the vehicle at time $t+\Delta t$ is $m(t)-b\Delta t$ where

$$\frac{dm}{dt} = -b, \quad (b \geq 0), \tag{4.2}$$

while the inertial velocity of the vehicle at time $t+\Delta t$ is ${}^{\mathcal{I}}\mathbf{v}(t)+\Delta^{\mathcal{I}}\mathbf{v}$ (see again, Fig. 4.1). Then the mass particle expended from the vehicle in the time increment Δt is given as

$$\Delta m = b\Delta t. \tag{4.3}$$

Therefore, the linear momentum of the system at time $t + \Delta t$ (that is, the linear momentum of the vehicle and the propellant mass Δm consumed in the time increment Δt) is given as

$${}^{\mathcal{I}}\mathbf{p}(t+\Delta t) = (m(t) - b\Delta t)({}^{\mathcal{I}}\mathbf{v}(t) + \Delta^{\mathcal{I}}\mathbf{v}) + b\Delta t^{\mathcal{I}}\mathbf{v}_{\Delta m}, \tag{4.4}$$

where ${}^{\mathcal{I}}\mathbf{v}_{\Delta m}$ is the inertial velocity of the mass particle Δm that has been expended in the time increment Δt. Suppose now that the velocity of the expended mass Δm *relative to the vehicle* is denoted ${}^{\mathcal{I}}\mathbf{v}_e$. It is noted that the quantity ${}^{\mathcal{I}}\mathbf{v}_e$ is called the *exhaust velocity* of the propellant. In terms of the exhaust vehicle, the inertial velocity of the mass particle Δm at time $t + \Delta t$ is given as

$${}^{\mathcal{I}}\mathbf{v}_{\Delta m} = {}^{\mathcal{I}}\mathbf{v}_e + {}^{\mathcal{I}}\mathbf{v} + \Delta^{\mathcal{I}}\mathbf{v}. \tag{4.5}$$

Consequently, the linear momentum of the system at time $t + \Delta t$ in Eq. (4.4) can be written as

$${}^{\mathcal{I}}\mathbf{p}(t+\Delta t) = (m(t) - b\Delta t)({}^{\mathcal{I}}\mathbf{v}(t) + \Delta^{\mathcal{I}}\mathbf{v}) + b\Delta t({}^{\mathcal{I}}\mathbf{v}_e + {}^{\mathcal{I}}\mathbf{v} + \Delta^{\mathcal{I}}\mathbf{v}). \tag{4.6}$$

Equation (4.6) can then be expanded to give

$$\begin{aligned} {}^{\mathcal{I}}\mathbf{p}(t+\Delta t) &= m(t)^{\mathcal{I}}\mathbf{v}(t) + m(t)\Delta^{\mathcal{I}}\mathbf{v} - b\Delta t^{\mathcal{I}}\mathbf{v}(t) - b\Delta t\Delta^{\mathcal{I}}\mathbf{v} \\ &\quad + b\Delta t^{\mathcal{I}}\mathbf{v}_e + b\Delta t^{\mathcal{I}}\mathbf{v}(t) + b\Delta t\Delta^{\mathcal{I}}\mathbf{v} \\ &= m(t)^{\mathcal{I}}\mathbf{v}(t) + m(t)\Delta^{\mathcal{I}}\mathbf{v} + b\Delta t^{\mathcal{I}}\mathbf{v}_e. \end{aligned} \tag{4.7}$$

Subtracting Eq. (4.1) from Eq. (4.7) gives

$${}^{\mathcal{I}}\mathbf{p}(t+\Delta t) - {}^{\mathcal{I}}\mathbf{p}(t) = m(t)\Delta^{\mathcal{I}}\mathbf{v} + b\Delta t^{\mathcal{I}}\mathbf{v}_e \tag{4.8}$$

Then, applying the principle of impulse and momentum to the vehicle, the change in linear momentum given in Eq. (4.8) is equal to the impulse $\mathbf{F}\Delta t$ applied over the time increment Δt which implies that

$${}^{\mathcal{I}}\mathbf{p}(t+\Delta t) - {}^{\mathcal{I}}\mathbf{p}(t) = m(t)\Delta^{\mathcal{I}}\mathbf{v} + b\Delta t^{\mathcal{I}}\mathbf{v}_e \approx \mathbf{F}\Delta t \tag{4.9}$$

Dividing Eq. (4.9) by Δt and taking the limit as $\Delta t \to 0$ in the inertial reference frame $\mathcal{I}$ gives

$$m\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\right) + b^{\mathcal{I}}\mathbf{v}_e = \mathbf{F} \tag{4.10}$$

which can be rearranged to give

$$\boxed{m\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\right) = -b^{\mathcal{I}}\mathbf{v}_e + \mathbf{F}.} \tag{4.11}$$

The quantity $-b^{\mathcal{I}}\mathbf{v}_e$ in Eq. (4.11) is called the *thrust force*. Observe that the exhaust velocity ${}^{\mathcal{I}}\mathbf{v}_e$ is in the direction *opposite* the velocity of the vehicle. Therefore, the quantity $-b^{\mathcal{I}}\mathbf{v}_e$ is in the direction of motion of the vehicle which should be the case if the vehicle is being propelled in a direction that increases its inertial speed. Suppose now that the thrust force is denoted $\mathbf{T}$, that is,

$$\mathbf{T} = -b^{\mathcal{I}}\mathbf{v}_e. \tag{4.12}$$

Then Eq. (4.11) can be written in terms of the force $\mathbf{T}$ as

$$\boxed{m\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\right) = \mathbf{T} + \mathbf{F}.} \tag{4.13}$$

It is noted that the resultant force applied to the vehicle remains separated into a thrust (propulsive) force and all other forces applied to the vehicle.

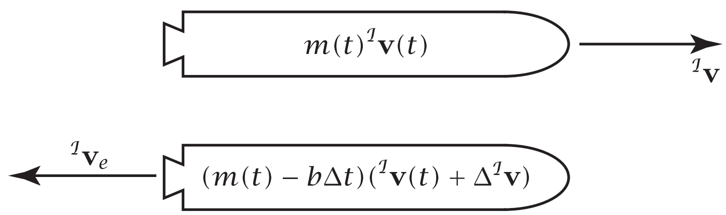


Figure 4.1 Schematic showing the linear momentum of the system consisting of a vehicle that is expending propellant during its motion. The linear momentum of the system is shown at two instants of time, t and $t + \Delta t$.

4.3 Solution of Rocket Equation

Consider now the case of Eq. (4.13) where $\mathbf{F} = \mathbf{0}$. Then Eq. (4.13) reduces to

$$m\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\right) = \mathbf{T}. \tag{4.14}$$

Now from the definition of the thrust force given in Eq. (4.12),

$$\|\mathbf{T}\| = T = \| - b^{\mathcal{I}}\mathbf{v}_e\| = bv_e, \tag{4.15}$$

where $v_e = \|{}^{\mathcal{I}}\mathbf{v}_e\|$. Solving Eq. (4.15) for b gives

$$b = \frac{T}{v_e}. \tag{4.16}$$

Then, substituting b from Eq. (4.16) into Eq. (4.2) gives

$$\dot{m} = -\frac{T}{v_e}. \tag{4.17}$$

Equation (4.14) can then be written as

$$m\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\right) = \mathbf{T} = -b^{\mathcal{I}}\mathbf{v}_e = -\frac{T}{v_e}{}^{\mathcal{I}}\mathbf{v}_e = T\mathbf{u}, \tag{4.18}$$

where

$$\mathbf{u} = -\frac{{}^{\mathcal{I}}\mathbf{v}_e}{v_e} \tag{4.19}$$

is the unit vector in the direction opposite the exhaust velocity. Dividing Eq. (4.18) by m gives

$$\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\right) = \mathbf{T} = -b\,{}^{\mathcal{I}}\mathbf{v}_e = -\frac{T}{v_e}{}^{\mathcal{I}}\mathbf{v}_e = \frac{T}{m}\mathbf{u}, \tag{4.20}$$

Suppose now that the unit vector $\mathbf{u}$ is fixed in the inertial frame $\mathcal{I}$. Because the only force applied to the vehicle lies in an inertially fixed direction, the inertial velocity also must lie in the that same inertially fixed direction which implies that

$${}^{\mathcal{I}}\mathbf{v} = v\mathbf{u}. \tag{4.21}$$

Then Eq. (4.20) can be re-written as

$$\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{v}\right) = \frac{{}^{\mathcal{I}}d}{dt}\left(v\mathbf{u}\right) = \frac{dv}{dt}\mathbf{u} = \frac{T}{m}\mathbf{u}, \tag{4.22}$$

where it is noted that ${}^{\mathcal{I}}d\mathbf{u}/dt = \mathbf{0}$. Equation (4.22) yields the scalar differential equation

$$\frac{dv}{dt} = \frac{T}{m}. \tag{4.23}$$

Now, solving Eq. (4.17) for T gives

$$T = -v_e\dot{m} = -v_e\frac{dm}{dt}. \tag{4.24}$$

Substituting T from Eq. (4.24) into Eq. (4.23) gives

$$\frac{dv}{dt} = -\frac{v_e\dot{m}}{m} = -\frac{v_e}{m}\frac{dm}{dt} \tag{4.25}$$

Therefore,

$$dv = -\frac{v_e}{m}dm \tag{4.26}$$

Integrating both sides of Eq. (4.26) gives

$$\int dv = v = \int -\frac{v_e}{m}dm = -v_e\ln|m| + C, \tag{4.27}$$

where C is a constant of integration. Now let t_0 be the initial time. Furthermore, let $v(t_0) = v_0$ and $m(t_0) = m_0$. Evaluating Eq. (4.27) at t_0 gives

$$v(t_0) = v_0 = -v_e\ln|m(t_0)| + C \tag{4.28}$$

from which the constant C is obtained as

$$C = v_0 + v_e\ln|m(t_0)| \tag{4.29}$$

Substituting C from Eq. (4.29) into Eq. (4.27) gives

$$v = -v_e\ln|m| + v_0 + v_e\ln|m(t_0)| \tag{4.30}$$

Equation (4.30) can be rearranged as

$$v - v_0 = v_e \ln \left| \frac{m_0}{m} \right| \tag{4.31}$$

Next, assume now that the thrust magnitude, T, is constant. Then integrating Eq. (4.17) from t_0 to t using the initial condition $m(t_0) = m_0$ gives

$$m(t) - m_0 = \int_{t_0}^{t} -\frac{T}{v_e} d\eta = -\frac{T}{v_e}(t - t_0). \tag{4.32}$$

which implies that

$$m(t) = m_0 - \frac{T}{v_e}(t - t_0). \tag{4.33}$$

Equation (4.31) can then be used to rewrite Eq. (4.31) as

$$v(t) - v_0 = v_e \ln \left| \frac{m_0}{m_0 - \dfrac{T}{v_e}(t - t_0)} \right| = v_e \ln \left| \frac{m_0}{m(t)} \right|. \tag{4.34}$$

Now, because the mass must be positive, the absolute value can be dropped in Eq. (4.34) which leads to

$$v(t) - v_0 = v_e \ln \left[\frac{m_0}{m(t)} \right]. \tag{4.35}$$

Setting $v = v(t)$, $m = m(t)$ and $\Delta v = v - v_0$, Eq. (4.35) can be written as

$$\boxed{\Delta v = v_e \ln \left(\frac{m_0}{m} \right).} \tag{4.36}$$

The quantity Δv in Eq. (4.36) is often referred to as the characteristic velocity or, more simply, "Delta-V".

Now, the exhaust speed v_e is typically written as the product of two quantities g_0 and I_{sp} as

$$v_e = g_0 I_{sp}, \tag{4.37}$$

where g_0 is the *standard Earth acceleration due to gravity at sea level* and has exact numeric value $g_0 = 9.80665 \text{ m} \cdot \text{s}^{-2}$ while I_{sp} is the specific impulse. Then, in terms of g_0 and I_{sp}, Eq. (4.36) can be written as

$$\boxed{\Delta v = g_0 I_{sp} \ln \left(\frac{m_0}{m} \right).} \tag{4.38}$$

Now, for completeness, from Eq. (4.23) it is seen that

$$\Delta v = \int_{t_0}^{t_f} \frac{T}{m} dt. \tag{4.39}$$

Equation (4.39) implies that the quantity

$$\int \frac{T}{m} dt \tag{4.40}$$

is an alternate expression for the characteristic velocity. Finally, it is seen that the characteristic velocity is equivalent to Δv and is a measure of the propellant consumed during a thrust maneuver.

4.4 Impulsive Thrust Approximation

An important approximation that is used in the propulsion of many spacecraft is the so called *impulsive thrust approximation*. Essentially, the impulsive thrust approximation assumes that an infinite amount of thrust can be applied to the spacecraft. While clearly it is not possible to apply an infinite amount of thrust, many spacecraft employ high-thrust chemical propulsion systems where an impulsive thrust approximation is reasonable. The basic idea behind the impulsive thrust approximation that, because the thrust is extremely large, the time of a high-thrust burn is very small in comparison to the time between burns. Thus, impulsive-thrust is an idealization where it is assumed that an infinite thrust can be applied instantaneously. It is noted that impulsive thrust is a poor approximation during the ascent phases of a rocket due to drag and gravity losses.

The impulsive-thrust approximation is obtained as follows. Consider a vehicle that is subject to only a thrust force **T**. Furthermore, consider a scalar version of Eq. (4.14) as

$$\frac{dv}{dt} = \frac{T}{m}. \tag{4.41}$$

Suppose now that the model for the thrust, T, is given as

$$T(t) = \left\{ \begin{array}{ccl} 0 & , & t \leq -\frac{\epsilon}{2}, \\ \frac{\hat{T}}{\epsilon} & , & -\frac{\epsilon}{2} \leq t \leq \frac{\epsilon}{2}, \\ 0 & , & t \geq \frac{\epsilon}{2}, \end{array} \right\}. \tag{4.42}$$

It is seen from Eq. (4.42) that

$$\int_{-\infty}^{\infty} T(t)dt = \hat{T}, \tag{4.43}$$

which is also the area under the thrust function curve. Suppose now that it is assumed that the value of ϵ approaches zero in such a manner that the area remains constant. The only way for the area to remain constant is if, in the limit as $\epsilon \to 0$, the thrust has the form

$$T(t) = \left\{ \begin{array}{ccl} 0 & , & t < 0, \\ \infty & , & t = 0, \\ 0 & , & t > 0. \end{array} \right\}. \tag{4.44}$$

Equation (4.44) is referred to as the *impulsive-thrust approximation* which can be equivalently written as

$$T = \hat{T}\delta(t), \tag{4.45}$$

where the function $\delta(t)$ is called the *Dirac delta function* and has the form

$$\delta(t) = \left\{ \begin{array}{ccl} 0 & , & t < 0, \\ \infty & , & t = 0, \\ 0 & , & t > 0. \end{array} \right\}. \tag{4.46}$$

The Dirac delta function is defined such that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1. \tag{4.47}$$

Thus the impulsive-thrust approximation is essentially a Dirac delta function scaled by a constant $\hat{T}$.

Suppose now that the impulsive-thrust approximation is applied to Eq. (4.41). Then,

$$\frac{dv}{dt} = \frac{\hat{T}}{m}\delta(t) \tag{4.48}$$

which implies that impulsive thrust is applied at time $t = 0$. Integrating both sides of Eq. (4.48) from $t = 0^-$ to $t = 0^+$ gives

$$\int_{0^-}^{0^+} \frac{dv}{dt}dt = \int_{0^-}^{0^+} \frac{\hat{T}}{m}\delta(t)dt. \tag{4.49}$$

Then, because the thrust is applied over a zero duration, Eq. (4.49) becomes

$$v(0^+) - v(0^-) = \Delta v = \int_{0^-}^{0^+} \frac{\hat{T}}{m}\delta(t)dt. \tag{4.50}$$

Then, from Eq. (4.38), the left-hand side of Eq. (4.50) is given as

$$\int_{0^-}^{0^+} \frac{\hat{T}}{m}\delta(t)dt = g_0 I_{sp} \ln\left(\frac{m^-}{m^+}\right) \tag{4.51}$$

which implies that

$$\boxed{\Delta v = g_0 I_{sp} \ln\left(\frac{m^-}{m^+}\right).} \tag{4.52}$$

Equation (4.52) states that for an impulsive-thrust approximation the velocity can be changed instantaneously.

Chapter 5

Impulsive Orbit Transfer

5.1 Introduction

The objective of this chapter is to begin studying the problem of orbit transfer utilizing many of the results that were developed in Chapters 1–3. The initial part of the study of orbit transfer focuses on the use of thrust (propulsive) force that consumes on board propellant (fuel) to modify the orbit of a spacecraft that would otherwise remain the same due to the fact that without thrust the motion of the spacecraft be Keplerian. As a first approximation, it is assumed that an infinite amount of thrust can be applied, thereby allowing for an instantaneous change in the inertial velocity of the spacecraft. A propulsion model that assumes infinite thrust is called an *impulsive* orbit transfer because the velocity can be changed without the passage of time. This chapter will describe the impulsive thrust model and justify why such a model is a good approximation for a variety of spacecraft. Then, using the impulsive thrust model, fundamental co-planar and non-co-planar transfers will be studied that transfer a spacecraft between either two circular orbits or between a circular orbit and an elliptic orbit. The fundamental co-planar transfer studied in this chapter is the Hohmann transfer with an extension of the Hohmann transfer called the bi-elliptic transfer. The the fundamental non-co-planar transfer is the non-co-planar extension of the Hohmann transfer where an impulse is applied to change the orbital inclination. Finally, conditions will be developed for intercept (where two objects arrive concurrently at the particular position) and rendezvous (where two vehicles arrive concurrently with the same position and inertial velocity).

5.2 Two-Impulse Transfer Between Co-Planar Circular Orbits

Consider now a transfer between two co-planar circular orbits. The radius of the initial orbit is r_1 while the radius of the terminal orbit is r_2. Furthermore, assume that the two orbits lie in the same plane (that is, the two orbits have the same orbital inclination). Suppose now that a spacecraft is in motion in the initial circular orbit of radius r_1 and the objective is to transfer the spacecraft from this initial circular orbit to the terminal circular orbit of radius r_2. As it turns out, provided that the two circular orbits are co-planar, the transfer can be accomplished using two impulses $\Delta^{\mathcal{I}}\mathbf{v}_1$ and $\Delta^{\mathcal{I}}\mathbf{v}_2$. Each of these impulses change the energy of the orbit by raising the periapsis, the apoapsis, or both the periapsis and the apoapsis of the orbit. Whether the periapsis, the apoapsis, or both apses are changed by an impulse depends upon the location where the impulse is applied. Because the transfer is co-planar, energy impulses occur in the orbit plane and are referred to an *energy change impulses*. The first impulse, $\Delta^{\mathcal{I}}\mathbf{v}_1$, places the spacecraft on an elliptic transfer orbit, where the elliptic transfer orbit has the same focus as that of the initial orbit. The second impulse, $\Delta^{\mathcal{I}}\mathbf{v}_2$, places the spacecraft in the terminal circular orbit. A schematic of a general two-impulse planar transfer between two circular orbits is shown in Fig. 5.1.

It is seen Fig. 5.1 that the transfer orbit (which in this case is an ellipse) must intersect both the initial orbit and the terminal orbit. As a result, all orbits whose periapsis radius is larger than r_1 (that is, any transfer orbit whose periapsis radius larger than the radius of the smaller circular orbit) and whose apoapsis radius is smaller than r_2 (that is, any transfer orbit whose periapsis radius is smaller than the radius of the larger circular orbit) are infeasible and are excluded as a possibility. Then, in terms of Eqs. (1.46) and (1.47), all feasible transfer orbits must satisfy the following conditions:

$$r_p = \frac{p}{1+e} \le r_1, \tag{5.1}$$

$$r_a = \frac{p}{1-e} \ge r_2. \tag{5.2}$$

Equations (5.1) and (5.2) can be re-written as

$$p \le r_1(1+e), \tag{5.3}$$

$$p \ge r_2(1-e). \tag{5.4}$$

5.3 The Hohmann Transfer

In 1925, Walter Hohmann published a monograph titled *Die Erreichbarkeit der Himmelsörpker* [English translation: *The Attainability of Celestial Bodies*]. Hohmann's conjecture at that time was that the minimum-fuel impulsive transfer between two co-planar circular orbits consisted of two impulses, where the first impulse is tangent to the initial orbit while the second impulse is tangent to the terminal orbit. The Hohmann transfer is shown in Fig. 5.2 where the radii of the initial and terminal circular orbits are r_1 and r_2, respectively, and $r_2 > r_1$* The first impulse, $\Delta^{\mathcal{I}}\mathbf{v}_1$, is then applied, changing orbit from the initial orbit to an elliptic transfer orbit whose periapsis is r_1 and whose apoapsis is r_2. The spacecraft then travels from periapsis to apoapsis of the transfer orbit. Upon reaching apoapsis of the transfer orbit a second impulse, $\Delta^{\mathcal{I}}\mathbf{v}_2$, is applied, changing the orbit from an elliptic transfer orbit with periapsis r_1 and apoapsis r_2 to a circular orbit of radius r_2. Because the first impulse changes the orbit from a circular orbit of radius r_1 to an elliptic orbit with periapsis r_1 and apoapsis r_2, the

*It is noted for completeness that for the case where $r_2 < r_1$ (that is, the case of a Hohmann transfer from a larger circular orbit to a smaller circular orbit) the transfer has the same form as it does for the case where $r_2 > r_1$, the only difference being that the directions of each of the two impulses are reversed from the case where $r_2 > r_1$.

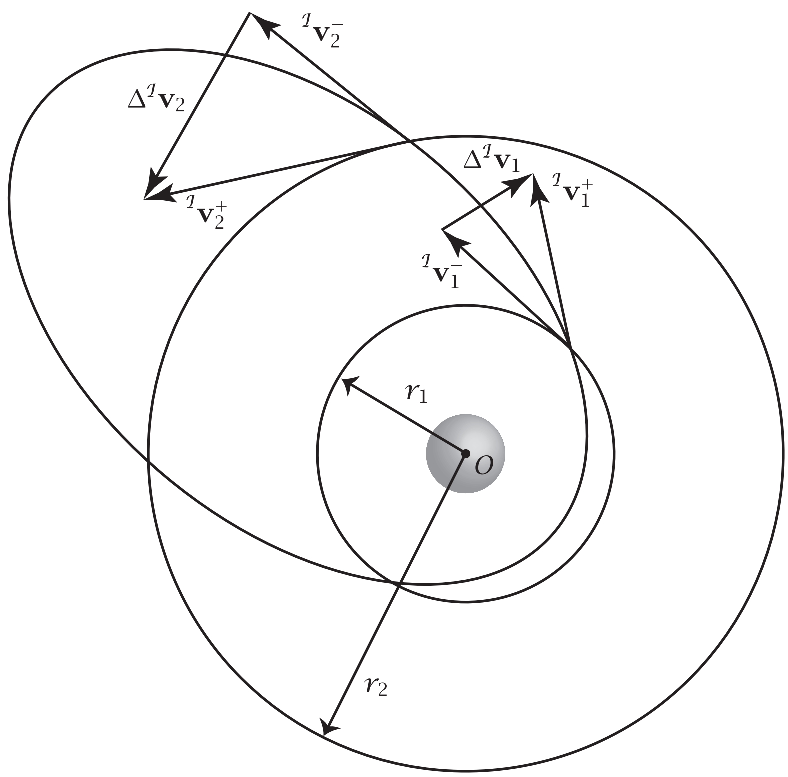


Figure 5.1 Schematic of a two-impulse transfer between two co-planar circular orbits of radii r_1 and r_2 where the transfer orbit is an ellipse with the same focus as that of both the initial and terminal orbits.

magnitude of the first impulse is given as

$$\Delta v_1 = \|\Delta^{\mathcal{I}}\mathbf{v}_1\| = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} - \sqrt{\frac{\mu}{r_1}} \tag{5.5}$$

where

$$\sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} > \sqrt{\frac{\mu}{r_1}} \tag{5.6}$$

because $a > r_1$. Now because the elliptic transfer orbit has periapsis and apoapsis radii equal to r_1 and r_2, respectively, the semi-major axis of the elliptic transfer orbit that results from the application of the first impulse is given as

$$a = \frac{r_1 + r_2}{2}. \tag{5.7}$$

Substituting a from Eq. (5.7) into (5.5), the magnitude of the first impulse is

$$\Delta v_1 = \sqrt{\frac{2\mu}{r_1} - \frac{2\mu}{r_1 + r_2}} - \sqrt{\frac{\mu}{r_1}}. \tag{5.8}$$

Equation (5.8) can be re-written as

$$\Delta v_1 = \sqrt{\frac{2\mu(r_1+r_2-r_1)}{r_1(r_1+r_2)}} - \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}}-1\right) \tag{5.9}$$

Similarly, the magnitude of the second impulse is given as

$$\Delta v_2 = \|\Delta^{\mathcal{I}}\mathbf{v}_2\| = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2}-\frac{\mu}{a}} \tag{5.10}$$

where it is noted that

$$\sqrt{\frac{\mu}{r_2}} > \sqrt{\frac{2\mu}{r_2}-\frac{\mu}{a}} \tag{5.11}$$

because $a < r_2$. Furthermore, it is noted for completeness that the semi-major axis used in Eq. (5.10) is the same as that used Eq. (5.5) because the second impulse is applied at the apoapsis of the elliptic transfer orbit. Substituting a from Eq. (5.7) into (5.10), the magnitude of the second impulse is

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2}-\frac{2\mu}{r_1+r_2}} \tag{5.12}$$

Then, following an approach similar to that used to simplify the expression for the first impulse, Eq. (5.12) can be re-written as

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu(r_1+r_2-r_2)}{r_2(r_1+r_2)}} = \sqrt{\frac{\mu}{r_2}}\left(1-\sqrt{\frac{2r_1}{r_1+r_2}}\right) \tag{5.13}$$

The total impulse for the Hohmann transfer is transfer a spacraft from an initial circular orbit of radius r_1 to a terminal circular orbit of radius $r_2 > r_1$ is the sum of the impulses given in Eqs. (5.9) and (5.13), that is

$$\Delta v_H = \Delta v_1 + \Delta v_2 = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}}-1\right) + \sqrt{\frac{\mu}{r_2}}\left(1-\sqrt{\frac{2r_1}{r_1+r_2}}\right) \tag{5.14}$$

Suppose now that the total impulse Δv_H is re-written in terms of the quantity

$$R = \frac{r_2}{r_1}, \tag{5.15}$$

where it is noted that R is the ratio of the radius of the larger orbit to the radius of the smaller orbit. Equation (5.14) can be re-written in terms of r_1 and R as follows. First, Eq. (5.14) can be re-written in terms of the ratio r_2/r_1 as

$$\Delta v_H = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2/r_1}{1+r_2/r_1}}-1\right) + \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{r_1}{r_2}}\left(1-\sqrt{\frac{2}{1+r_2/r_1}}\right). \tag{5.16}$$

Then, substituting R from Eq. (5.15) into (5.16) gives

$$\Delta v_H = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2R}{1+R}}-1\right) + \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{1}{R}}\left(1-\sqrt{\frac{2}{1+R}}\right). \tag{5.17}$$

Then, factoring out the common factor $\sqrt{\mu/r_1}$ from each term in Eq. (5.17), Δv_H can be written as

$$\Delta v_H = \sqrt{\frac{\mu}{r_1}}\left[\left(\sqrt{\frac{2R}{1+R}}-1\right)+\sqrt{\frac{1}{R}}\left(1-\sqrt{\frac{2}{1+R}}\right)\right]. \tag{5.18}$$

Suppose now that we define the initial circular speed as

$$v_{c1} = \sqrt{\frac{\mu}{r_1}}. \tag{5.19}$$

Normalizing the total impulse for the Hohmann transfer by v_{c1} gives

$$\frac{\Delta v_H}{v_{c1}} = \left(\sqrt{\frac{2R}{1+R}}-1\right)+\sqrt{\frac{1}{R}}\left(1-\sqrt{\frac{2}{1+R}}\right) \tag{5.20}$$

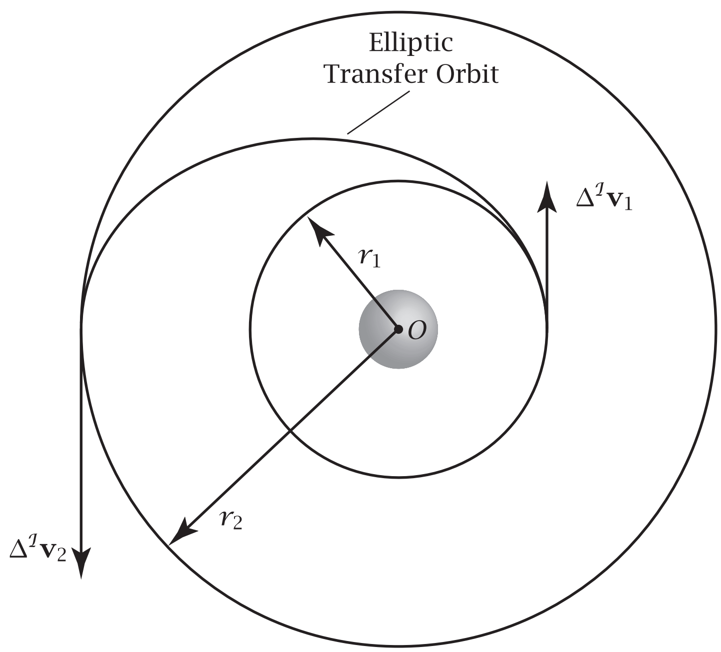


Figure 5.2 Schematic of a Hohmann transfer between two co-planar circular orbits of radii r_1 and r_2. The transfer orbit is tangent at its periapsis to the initial orbit and is tangent at its apoapsis to the terminal orbit.

5.4 Bi-Elliptic Transfer

A commonly used extension of the Hohmann transfer a three-impulse transfer known as the bi-elliptic transfer. Consider again a co-planar transfer from an initial circular orbit of radius r_1 to a terminal circular orbit of radius r_2. The key difference between the bi-elliptic transfer and the Hohmann transfer is that the bi-elliptic transfer consists of three impulses and two elliptic transfer orbits while the Hohmann transfer consists of two impulses and one elliptic transfer orbit. Similar to the Hohmann transfer, however, all three impulses that define the bi-elliptic transfer are applied tangentially at either periapsis or apoapsis at either the initial or terminal orbit or one of the two transfer orbits.

The first impulse, applied tangentially at a point on the initial circular orbit, places the spacecraft onto a first elliptic transfer orbit whose apoapsis, r_i, is *larger* than r_2 (that is, $r_i > r_2$ where r_2 is the radius of the terminal circular orbit). The second impulse, applied at the apoapsis of the first elliptic transfer orbit and places the spaceacraft onto a second elliptic transfer orbit whose periapsis is equal to the radius of the terminal circular orbit (that is, the periapsis of the second transfer orbit is equal to r_2. The third impulse, applied at the periapsis of the second transfer orbit, then reduces the apoapsis from r_i to the radius, r_2, of the terminal orbit, thereby placing the spacecraft in the terminal circular orbit of radius r_2. A schematic of the bi-elliptic transfer is shown in Fig. 5.3, where the radii of the initial and terminal circular orbits are r_1 and r_2, respectively, $r_2 > r_1$, and the apoapsis of the first elliptic transfer orbit is r_i[†] The spacecraft is an initial circular orbit of radius r_1. The first impulse, $\Delta^{\mathcal{I}}\mathbf{v}_1$, is then applied, changing the orbit from the initial circular orbit of radius r_1 to an elliptic transfer orbit whose periapsis is r_1 and whose apoapsis is r_{a1}. The spacecraft then travels from periapsis to apoapsis of the first elliptic transfer orbit. Upon reaching apoapsis of the first elliptic transfer orbit a second impulse, $\Delta^{\mathcal{I}}\mathbf{v}_2$, is applied, changing the orbit from the first elliptic transfer orbit with periapsis r_1 and apoapsis r_{a1} to a second elliptic transfer orbit with periapsis r_2 and apoapsis r_{a1}. The spacecraft then travels from apoapsis to periapsis of the second elliptic transfer orbit. Upon reaching periapsis of the second elliptic transfer orbit a third impulse, $\Delta^{\mathcal{I}}\mathbf{v}_3$, is applied, changing the orbit from the second elliptic transfer orbit with periapsis r_2 and apoapsis r_{a1} to the terminal circular orbit of radius r_2. It is noted that, because the first two impulses increase one of the apses (specifically, the first impulse increases apoapsis while the second impulse increases periapsis) while the third impulse decreases the apoapsis of the second transfer elliptic orbit, the first two impulses are posigrade while the third impulse is retrograde. Similar to the approach developed to obtain the total impulse for the Hohmann transfer, because each impulse of the bi-elliptic transfer is applied tangentially, all vector quantities can be replaced by scalars in the derivation that follows.

The first impulse of the bi-elliptic transfer, applied tangentially at the initial circular orbit of radius r_1, increases apoapsis and places the spacecraft onto an elliptic orbit with periapsis radius r_1 and apoapsis radius r_i. Therefore, this first impulse is given as

$$\Delta v_1 = \|\Delta^{\mathcal{I}}\mathbf{v}_1\| = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a_1}} - \sqrt{\frac{\mu}{r_1}}, \tag{5.21}$$

where $a_1 = (r_1 + r_i)/2$ is the semi-major axis of the first elliptic transfer orbit. Thus, Δv_1 can be written as

$$\Delta v_1 = \sqrt{\frac{2\mu}{r_1} - \frac{2\mu}{r_1 + r_i}} - \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_i}{r_1 + r_i}} - 1\right). \tag{5.22}$$

[†]It is noted for completeness that for the case where $r_2 < r_1$ (that is, the case of a transfer from a larger circular orbit to a smaller circular orbit) the transfer has the same form as it does for the case where $r_2 > r_1$, the only difference being that the directions of each of the three impulses are reversed from the case where $r_2 > r_1$.

The second impulse of the bi-elliptic transfer, applied tangentially at the apoapsis of the first elliptic transfer orbit, increases periapsis and places the spacecraft onto a second elliptic orbit with periapsis radius r_2 and apoapsis radius r_i. Therefore, this second impulse is given as

$$\Delta v_2 = \|\Delta^{\mathcal{I}}\mathbf{v}_2\| = \sqrt{\frac{2\mu}{r_i} - \frac{\mu}{a_2}} - \sqrt{\frac{2\mu}{r_i} - \frac{\mu}{a_1}}, \tag{5.23}$$

where $a_2 = (r_2 + r_i)/2$ is the semi-major axis of the second elliptic transfer orbit. Thus, Δv_2 can be written as

$$\begin{aligned}\Delta v_2 = \|\Delta^{\mathcal{I}}\mathbf{v}_2\| &= \sqrt{\frac{2\mu}{r_i} - \frac{2\mu}{r_2 + r_i}} - \sqrt{\frac{2\mu}{r_i} - \frac{2\mu}{r_1 + r_i}} \\ &= \sqrt{\frac{\mu}{r_i}}\left(\sqrt{\frac{2r_2}{r_2 + r_i}} - \sqrt{\frac{2r_1}{r_1 + r_i}}\right).\end{aligned} \tag{5.24}$$

The third impulse of the bi-elliptic transfer, applied tangentially at the periapsis of the second elliptic transfer orbit, decreases apoapsis and places the spacecraft into the terminal circular orbit of radius r_2. Noting that the third impulse is retrograde (in that it decreases apoapsis), this third impulse is given as

$$\Delta v_3 = \|\Delta^{\mathcal{I}}\mathbf{v}_3\| = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a_2}} - \sqrt{\frac{\mu}{r_2}} \tag{5.25}$$

Consequently, Δv_3 can be written as

$$\Delta v_3 = \|\Delta^{\mathcal{I}}\mathbf{v}_3\| = \sqrt{\frac{2\mu}{r_2} - \frac{2\mu}{r_2 + r_i}} - \sqrt{\frac{\mu}{r_2}} = \sqrt{\frac{\mu}{r_2}}\left(\sqrt{\frac{2r_i}{r_2 + r_i}} - 1\right). \tag{5.26}$$

The total impulse for the bi-elliptic transfer is then obtained by adding the results of Eqs. (5.22), (5.24), and (5.26) as

$$\Delta v_{BE} = \Delta v_1 + \Delta v_2 + \Delta v_3. \tag{5.27}$$

Suppose now that we define the quantity

$$S = \frac{r_i}{r_2}. \tag{5.28}$$

The the first impulse, given in Eq. (5.22) can then be written as

$$\begin{aligned}\Delta v_1 &= \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_i}{r_1 + r_i}} - 1\right) = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_i/r_2}{r_1/r_2 + r_i/r_2}} - 1\right) \\ &= \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2S}{1/R + S}} - 1\right) = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2RS}{1 + RS}} - 1\right).\end{aligned} \tag{5.29}$$

Next, the second impulse, given in Eq. (5.24), can be written as

$$\begin{aligned}\Delta v_2 &= \sqrt{\frac{\mu}{r_i}}\left(\sqrt{\frac{2r_2}{r_2 + r_i}} - \sqrt{\frac{2r_1}{r_1 + r_i}}\right) = \sqrt{\frac{\mu}{r_1}\frac{r_1}{r_2}\frac{r_2}{r_i}}\left(\sqrt{\frac{2}{1 + r_i/r_2}} - \sqrt{\frac{2r_1/r_2}{r_1/r_2 + r_i/r_2}}\right) \\ &= \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{1}{RS}}\left(\sqrt{\frac{2}{1 + S}} - \sqrt{\frac{2/R}{1/R + S}}\right) = \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{1}{RS}}\left(\sqrt{\frac{2}{1 + S}} - \sqrt{\frac{2}{1 + RS}}\right).\end{aligned} \tag{5.30}$$

Finally, the third impulse, given in Eq. (5.26), can be written as

$$\begin{aligned}\Delta v_3 &= \sqrt{\frac{\mu}{r_2}}\left(\sqrt{\frac{2r_i}{r_2 + r_i}} - 1\right) = \sqrt{\frac{\mu}{r_1}\frac{r_1}{r_2}}\left(\sqrt{\frac{2r_i/r_2}{1 + r_i/r_2}} - 1\right) \\ &= \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{1}{R}}\left(\sqrt{\frac{2S}{1+S}} - 1\right) = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2S}{R+RS}} - \sqrt{\frac{1}{R}}\right)\end{aligned} \tag{5.31}$$

Adding Eqs. (5.29)–(5.31) and factoring out the common factoring of $\sqrt{\mu/r_1}$, the total impulse for the bi-elliptic transfer in terms of the ratios R and S is given as

$$\Delta v_{BE} = \sqrt{\frac{\mu}{r_1}}\left[\left(\sqrt{\frac{2RS}{1+RS}} - 1\right) + \sqrt{\frac{1}{RS}}\left(\sqrt{\frac{2}{1+S}} - \sqrt{\frac{2}{1+RS}}\right) + \left(\sqrt{\frac{2S}{R+RS}} - \sqrt{\frac{1}{R}}\right)\right]. \tag{5.32}$$

Recal now from Eq. (5.19) that the $v_{c1} = \sqrt{\mu/r_1}$ is the initial circular speed. Normalizing the total impulse of the bi-elliptic transfer by v_{c1} gives

$$\frac{\Delta v_{BE}}{v_{c1}} = \left[\left(\sqrt{\frac{2RS}{1+RS}} - 1\right) + \sqrt{\frac{1}{RS}}\left(\sqrt{\frac{2}{1+S}} - \sqrt{\frac{2}{1+RS}}\right) + \left(\sqrt{\frac{2S}{R+RS}} - \sqrt{\frac{1}{R}}\right)\right]. \tag{5.33}$$

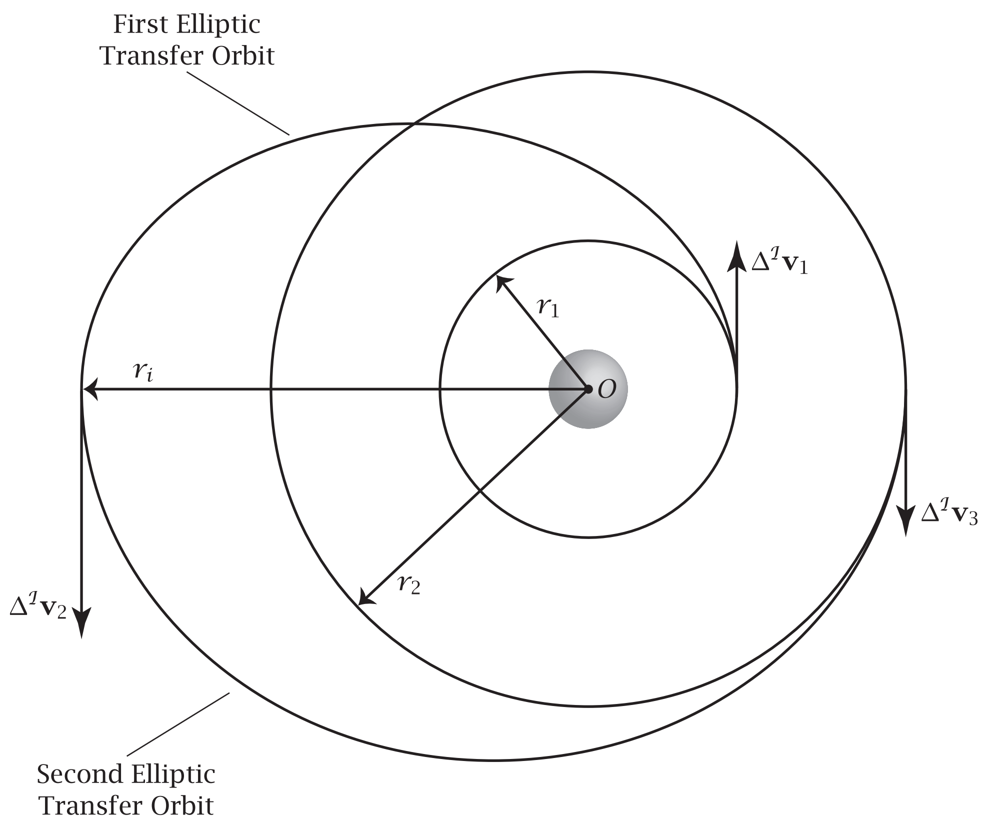


Figure 5.3 Schematic of a Bi-Elliptic transfer between two co-planar circular orbits of radii r_1 and r_2. The transfer consists of three impulses and two transfer orbits. The first impulse places the spacecraft onto a first elliptic transfer orbit whose apoapsis is larger than the radius of the terminal circular orbit. The second impulse places the spacecraft onto a second elliptic transfer orbit whose periapsis is equal to the radius of the terminal circular orbit. The third impulse reduces the apoapsis of the second transfer orbit to the radius of the terminal circular orbit, thereby placing the spacecraft into the final circular orbit of radius r_2.

5.5 Bi-Parabolic Transfer

Consider now a limiting case of a bi-elliptic transfer called the *bi-parabolic transfer*. The bi-parabolic transfer is obtained by letting $r_i \to \infty$ which implies equivalently that $S \to \infty$. First, normalizing the impulses in Eqs. (5.29)–(5.31) by $v_{c1} = \sqrt{\mu/r_1}$ gives

$$\begin{aligned}
\frac{\Delta v_1}{v_{c1}} &= \left(\sqrt{\frac{2RS}{1+RS}} - 1\right), && (5.34)\\
\frac{\Delta v_2}{v_{c1}} &= \sqrt{\frac{1}{RS}}\left(\sqrt{\frac{2}{1+S}} - \sqrt{\frac{2}{1+RS}}\right), && (5.35)\\
\frac{\Delta v_3}{v_{c1}} &= \left(\sqrt{\frac{2S}{R+RS}} - \sqrt{\frac{1}{R}}\right). && (5.36)
\end{aligned}$$

Next, Eqs. (5.34)–(5.36) can be re-written as

$$\begin{aligned}
\frac{\Delta v_1}{v_{c1}} &= \left(\sqrt{\frac{2R}{1/S+R}} - 1\right), && (5.37)\\
\frac{\Delta v_2}{v_{c1}} &= \sqrt{\frac{1}{RS}}\left(\sqrt{\frac{2/S}{1/S+1}} - \sqrt{\frac{2/S}{1/S+R}}\right), && (5.38)\\
\frac{\Delta v_3}{v_{c1}} &= \left(\sqrt{\frac{2}{R/S+R}} - \sqrt{\frac{1}{R}}\right). && (5.39)
\end{aligned}$$

Then, taking $\lim S \to \infty$ in Eqs. (5.37)–(5.39) gives

$$\begin{aligned}
\frac{\Delta v_1}{v_{c1}} &= \sqrt{2} - 1, && (5.40)\\
\frac{\Delta v_2}{v_{c1}} &= 0, && (5.41)\\
\frac{\Delta v_3}{v_{c1}} &= \sqrt{\frac{1}{R}}\left(\sqrt{2} - 1\right). && (5.42)
\end{aligned}$$

Therefore, the total impulse for the bi-parabolic transfer is given as

$$\Delta v_{BP} = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{2} - 1\right)\left(1 + \sqrt{\frac{1}{R}}\right). \tag{5.43}$$

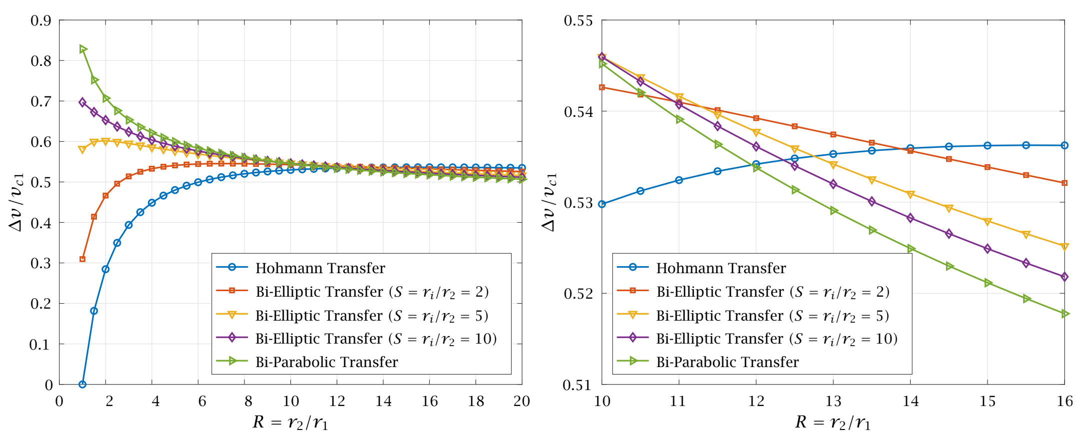
Equivalently, the total impulse of the bi-parabolic transfer normalized by the initial circular speed $v_{c1} = \sqrt{\mu/r_1}$ is given as

$$\frac{\Delta v_{BP}}{v_{c1}} = \left(\sqrt{2} - 1\right)\left(1 + \sqrt{\frac{1}{R}}\right). \tag{5.44}$$

It is seen that the bi-parabolic transfer is a two-impulse limiting case of the three-impulse bi-elliptic transfer. It is noted, however, that the bi-parabolic transfer is physically unrealizable because the transfer consists of outbound and inbound parabolic trajectories for which the spacecraft needs to travel an infinite distance (thus requiring an infinite amount of time). While the bi-parabolic transfer is physically unrealizable, it is a good approximation of a bi-elliptic transfer with a sufficiently large transfer orbit apoapsis r_i because when r_i is sufficiently large the second impulse of the bi-elliptic transfer is small and can be neglected.

5.6 Comparison of Co-Planar Impulsive Transfers

A comparison of the performance of the co-planar Hohmann, bi-elliptic, and bi-parabolic transfers described in Sections 5.3–5.5 is now made. Figure 5.4(a) shows the normalized impulse (that is, the impulse normalized by the initial circular speed $v_{c1} = \sqrt{\mu/r_1}$) for the Hohmann transfer, the bi-elliptic transfer with $S = (2, 5, 10)$, and the bi-parabolic transfer as a function of the ratio $R = r_2/r_1$ between the two circular orbits. It is seen that for smaller values of R the Hohmann transfer outperforms either the bi-elliptic or the bi-parabolic transfer. Note from Fig. 5.4(b), however, that beyond critical values of R the bi-elliptic and bi-parabolic transfers outperform the Hohmann transfer. Specifically, these critical values are as follows. First, for $1 < R < 11.94$ the Hohmann transfer is the absolute minimum impulse transfer between two circular orbits. For $R > 11.94$ the bi-parabolic transfer outperforms the Hohmann transfer. Finally, for $R > 15.58$ any bi-elliptic transfer with $S > 1$ outperforms the Hohmann transfer. Finally, for $11.94 < R < 15.58$ the bi-elliptic is more economical than the Hohmann transfer only if the apoapsis r_i of the two elliptic transfer orbits is sufficiently large.



(a) $\Delta v/v_{c1}$ vs. $R = r_2/r_1$.

(b) Enlarged view of $\Delta v/v_{c1}$ vs. $R = r_2/r_1$.

Figure 5.4 Comparisons of normalized total impulse, $\Delta v/v_{c1}$, vs. $R = r_2/r_1$ for the Hohmann transfer, bi-elliptic transfer with $S = r_i/r_2 = (2, 5, 10)$, and bi-parabolic transfer.

5.7 Non-Co-Planar Impulsive Transfer

Until now the focus has been on co-planar circle-to-circle impulsive orbital transfer. In many applications of orbital transfer, however, it is necessary to change the inclination of the orbit. Inclination change on a elliptic orbit has one of two forms. The first form is an impulse that changes only the inclination of the orbit without changing the energy of the orbit (that is, neither the size nor the shape of the orbit is changed). The second form of an impulsive inclination change combines an inclination change impulse with an impulse that changes the energy of the orbit (that is, the part of the impulse that changes the energy of the orbit changes the size and shape of the orbit). The next two sections focus on both types of inclination change impulses.

5.7.1 Impulse That Changes Orbital Plane

Suppose we consider the problem of using a single impulsive maneuever $\Delta^{\mathcal{I}}\mathbf{v}$ applied at a point along a circular orbit in order to change the plane of the orbit by an angle θ. Suppose further that the velocity of the spacecraft the instants before the impulse $\Delta^{\mathcal{I}}\mathbf{v}$ is applied is ${}^{\mathcal{I}}\mathbf{v}^-$ while the velocity of the spacecraft the instant after the impulse is applied is denoted ${}^{\mathcal{I}}\mathbf{v}^+$. Because objective is to rotate the plane of the orbit, the magnitudes of ${}^{\mathcal{I}}\mathbf{v}^-$ and ${}^{\mathcal{I}}\mathbf{v}^+$ are the same, that is, $\|{}^{\mathcal{I}}\mathbf{v}^-\| = \|{}^{\mathcal{I}}\mathbf{v}^-\| = v$. Figure 5.5(a) provides a schematic of an impulse $\Delta^{\mathcal{I}}\mathbf{v}$ applied purely for the purpose of changing the plane of the circular orbit. It is seen from Fig. 5.5(a) that the pre-impulse and post-impulse velocities together with the impulse $\Delta^{\mathcal{I}}\mathbf{v}$ form an isosceles triangle where the pre-impulse and post-impulse velocities are the two sides of the triangle with equal-length sides. Consequently, bisecting the angle θ as shown in Fig. 5.5(b) creates two right-triangles where the hypotenuse has a magnitude v while the side opposite the angle $\theta/2$ has a magnitude $\Delta v/2 = \|\Delta^{\mathcal{I}}\mathbf{v}_i\|/2$. Thus, the magnitude v and $\Delta v/2$ are related to the angle $\theta/2$ as

$$\sin\frac{\theta}{2} = \frac{\Delta v/2}{v}. \tag{5.45}$$

Solving Eq. (5.45) for Δv_i gives

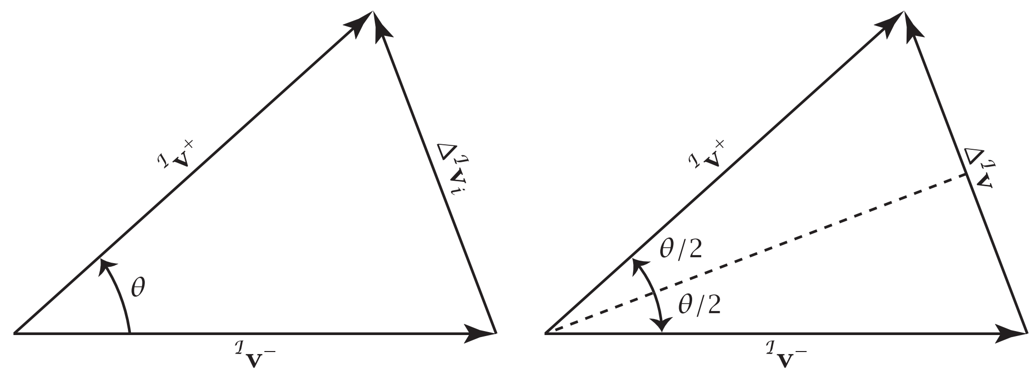
$$\Delta v = 2v\sin\frac{\theta}{2}. \tag{5.46}$$

An impulse of the form given in Eq. (5.46) is called a *cranking impulse* and the process of changing the plane of the orbit using such an impulse is called *orbit cranking*. It is seen from Eq. (5.46) that, for a given amount of rotation of the orbital plane θ, the impulse Δv is proportional to the speed v.

Now it is noted that the angle θ of a cranking impulse as given in Eq. (5.46) that defines the amount by which the orbit plane is rotated is in general *not* equal to the change in the orbital inclination. In other words, in general it is not the case that $\theta = \Delta i$, where Δi is the amount of the inclination change. In order to see that in general $\theta \neq \Delta i$, consider the Fig. 5.6 that shows two non-co-planar circular orbits of the same size (that is, $r_1 = r_2 = r$) with inclinations i_1 and i_2, respectively. Consider further that the orbits have longitudes of ascending nodes Ω_1 and Ω_2, respectively. The specific angular momenta of the first and second orbits are given in planet-centered inertial coordinates as

$$\begin{aligned} {}^{\mathcal{I}}\mathbf{h}_1 &= h(\sin i_1 \sin\Omega_1\mathbf{I}_x - \sin i_1\cos\Omega_1\mathbf{I}_y + \cos i_1\mathbf{I}_z), \\ {}^{\mathcal{I}}\mathbf{h}_2 &= h(\sin i_2 \sin\Omega_2\mathbf{I}_x - \sin i_2\cos\Omega_2\mathbf{I}_y + \cos i_2\mathbf{I}_z), \end{aligned} \tag{5.47}$$

where it is noted that $\|{}^{\mathcal{I}}\mathbf{h}_1\| = \|{}^{\mathcal{I}}\mathbf{h}_2\| = h$ because $r_1 = r_2$. Taking the scalar product of



(a) Inclination change impulse, $\Delta^{\mathcal{I}}\mathbf{v}_i$ (b) Bisection of inclination change impulse, $\Delta^{\mathcal{I}}\mathbf{v}_i$

Figure 5.5 Impulse, $\Delta^{\mathcal{I}}\mathbf{v}$, applied at either periapsis or apoapsis of an orbit in order to change the orbital inclination by an angle Δi.

$^{\mathcal{I}}\mathbf{h}_1$ and $^{\mathcal{I}}\mathbf{h}_2$ and simplifying the result gives

$$\begin{aligned}
^{\mathcal{I}}\mathbf{h}_1 \cdot {}^{\mathcal{I}}\mathbf{h}_2 &= h^2(\sin i_1 \sin\Omega_1 \sin i_2 \sin\Omega_2 + \sin i_1 \cos\Omega_1 \sin i_2 \cos\Omega_2 + \cos i_1 \cos i_2) \\
&= h^2(\sin i_1 \sin i_2(\sin\Omega_1 \sin\Omega_2 + \cos\Omega_1 \cos\Omega_2) + \cos i_1 \cos i_2) \\
&= h^2(\sin i_1 \sin i_2 \cos(\Omega_2 - \Omega_1) + \cos i_1 \cos i_2) \\
&= h^2(\sin i_1 \sin i_2 \cos\Delta\Omega + \cos i_1 \cos i_2) \\
&= h^2(\cos i_1 \cos i_2 + \cos\Delta\Omega \sin i_1 \sin i_2),
\end{aligned} \tag{5.48}$$

where $\Delta\Omega = \Omega_2 - \Omega_1$. But from the definition of the scalar product, the scalar product of the two angular momenta is also given as

$$^{\mathcal{I}}\mathbf{h}_1 \cdot {}^{\mathcal{I}}\mathbf{h}_2 = h^2 \cos\theta \tag{5.49}$$

Setting the results of Eqs. (5.48) and (5.49) equal gives

$$\cos\theta = \cos i_1 \cos i_2 + \cos\Delta\Omega \sin i_1 \sin i_2 \tag{5.50}$$

Suppose now that $\Delta\Omega = 0$. Then Eq. (5.50) simplifies to

$$\cos\theta = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 = \cos(i_2 - i_1) = \cos\Delta i, \tag{5.51}$$

where $\Delta i = i_2 - i_1$. Equation (5.51) implies that $\theta = i_2 - i_1$ *only* in the case where the longitudes of the ascending nodes of the two orbits are equal. Furthermore, in the case where $\Omega_2 = \Omega_1$, Eq. (5.51) implies that the rotation of the orbital plane by the angle θ (which, as stated, is equal to Δi when $\Omega_2 = \Omega_1$) must be performed at the line of nodes. In other words, the angle θ that defines the amount by which the plane of a circular orbit is changed by a cranking impulse $\Delta^{\mathcal{I}}\mathbf{v}$ is equal to the change in the orbital inclination only if the longitudes of the ascending node of both orbits are the same and the cranking impulse is applied at the ascending node.

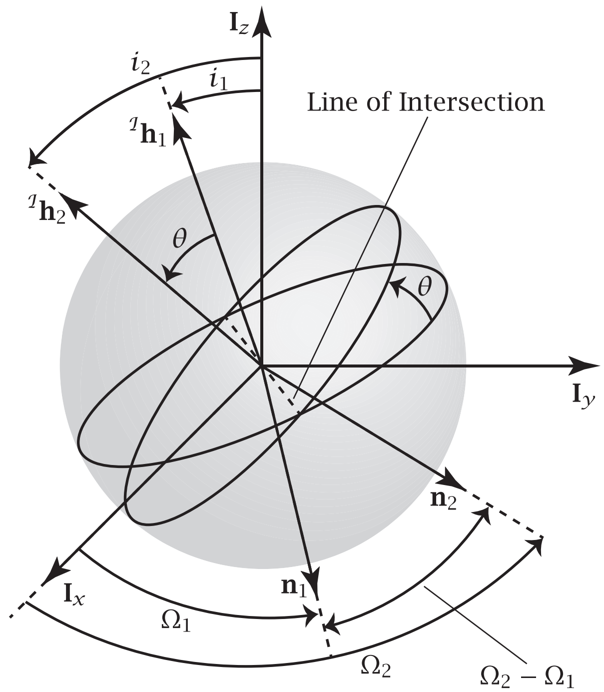


Figure 5.6 Schematic showing two orbits with different inclinations and different longitudes of ascending node.

5.7.2 Impulse That Combines Orbit Plane Rotation and Energy Change

Consider now an impulse $\Delta^{\mathcal{I}}\mathbf{v}$ that both rotates the orbital plane and changes the energy of the orbit. Let ${}^{\mathcal{I}}\mathbf{v}^-$ and ${}^{\mathcal{I}}\mathbf{v}^+$ be the inertial velocity of the spacecraft the instant before and the instant after the impulse $\Delta^{\mathcal{I}}\mathbf{v}$ is applied. Furthermore, because the impulse $\Delta^{\mathcal{I}}\mathbf{v}$ both rotates the orbit plane and changes the energy, the pre-impulse and post-impulse speed will be different, that is, $v^- = \|{}^{\mathcal{I}}\mathbf{v}^-\| \neq \|{}^{\mathcal{I}}\mathbf{v}^+\| = v^+$. Figure 5.7 shows the velocity of the spacecraft the instant before and the instant after the impulse is applied along with the impulse itself. It is seen from Fig. 5.7 that the impulse $\Delta^{\mathcal{I}}\mathbf{v}$ is given as

$$\Delta^{\mathcal{I}}\mathbf{v} = {}^{\mathcal{I}}\mathbf{v}^+ - {}^{\mathcal{I}}\mathbf{v}^-. \tag{5.52}$$

Noting that the angle between ${}^{\mathcal{I}}\mathbf{v}^-$ and ${}^{\mathcal{I}}\mathbf{v}^+$ is θ, the magnitude of the impulse $\Delta^{\mathcal{I}}\mathbf{v}$ is obtained from the law of cosines as

$$\boxed{\Delta v^2 = \|\Delta^{\mathcal{I}}\mathbf{v}\|^2 = (v^-)^2 + (v^+)^2 - 2v^-v^+\cos\theta.} \tag{5.53}$$

Figure 5.7 shows the impulse $\Delta^{\mathcal{I}}\mathbf{v}$ that simultaneously rotates both the orbit plane and the energy. It is noted that the orbital energy the instant before the application of the impulse $\Delta^{\mathcal{I}}\mathbf{v}$ is based on the pre-impulse orbital speed v^- while the orbital energy the instant after the application of the impulse $\Delta^{\mathcal{I}}\mathbf{v}$ is based on the post-impulse orbital speed v^+. Finally, it is emphasized again that the impulse $\Delta^{\mathcal{I}}\mathbf{v}$ shown in Fig. 5.7 is a combined plane change and energy change impulse that corresponds to that given in Eq. (5.53).

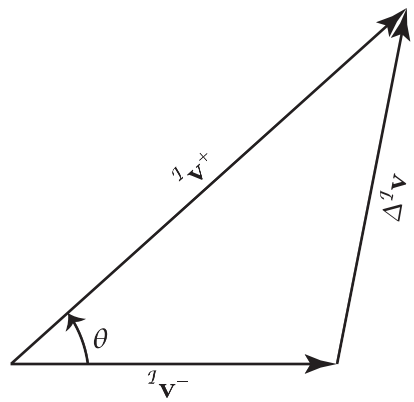


Figure 5.7 $\Delta^{\mathcal{I}}\mathbf{v}$, that rotates orbit plane and changes energy simultaneously.

5.7.3 Two-Impulse Transfer Between Non-Co-Planar Circular Orbits

Sections 5.7.1 and 5.7.2 provide elements that can be used to construct a two-impulse transfer between two non-co-planar circular orbits with radii r_1 and r_2, respectively, where it is assumed that $r_2 > r_1$. Note first that if the planes of two orbits have a common point (in this case, both orbits share a common focus), then the two orbits must intersect in a line. The unit vector along the line of intersection between the two orbits is determined as follows. First, because the specific angular momentum of each orbit lies normal to the orbit plane of the respective orbit, the vector product $^{\mathcal{I}}\mathbf{h}_1$ and $^{\mathcal{I}}\mathbf{h}_2$, $^{\mathcal{I}}\mathbf{h}_1 \times {}^{\mathcal{I}}\mathbf{h}_2$ must lie in the plane of both orbits because $^{\mathcal{I}}\mathbf{h}_1 \times {}^{\mathcal{I}}\mathbf{h}_2$ is orthogonal to both $^{\mathcal{I}}\mathbf{h}_1$ and $^{\mathcal{I}}\mathbf{h}_2$. Therefore, the line of intersection in the direction of $^{\mathcal{I}}\mathbf{h}_1 \times {}^{\mathcal{I}}\mathbf{h}_2$. Recalling the assumption that the two orbits are non-co-planar, the orbital inclinations of the two orbits are different (that is, $i_2 \neq i_1$). Then, the unit vector along the line of intersection between the two orbits, denoted $\boldsymbol{\ell}$, is defined as the unit vector

$$\boldsymbol{\ell} = \frac{^{\mathcal{I}}\mathbf{h}_1 \times {}^{\mathcal{I}}\mathbf{h}_2}{\|^{\mathcal{I}}\mathbf{h}_1 \times {}^{\mathcal{I}}\mathbf{h}_2\|}. \tag{5.54}$$

It is noted that $\boldsymbol{\ell}$ will be undefined in the case where the two orbits are co-planar because $^{\mathcal{I}}\mathbf{h}_1 \times {}^{\mathcal{I}}\mathbf{h}_2$ will be zero for the case of co-planar orbits.

Using the line of intersection $\boldsymbol{\ell}$ as defined in Eq. (5.54) as a starting point, consider now a two-impulse orbit transfer between two non-co-planar circular orbits. First, because any feasible orbit transfer must intersection both orbits, it is necessary that any impulse that transfers the spacecraft between the two orbits occur at a position on the orbit that lies along the line of intersection between the two orbits. This last fact leads to the following construction of a two-impulse orbital transfer between non-co-planar orbits. First, compute the line of intersection $\boldsymbol{\ell}$ as given in Eq. (5.54) using the angular momenta $^{\mathcal{I}}\mathbf{h}_1$ and $^{\mathcal{I}}\mathbf{h}_2$ associated with the initial and terminal orbits. Next, let $\mathbf{r}_1$ be the position on the initial orbit where the first impulse $\Delta^{\mathcal{I}}\mathbf{v}$, is applied. It is noted that the position $\mathbf{r}_1$ is given as

$$\mathbf{r}_1 = r_1\boldsymbol{\ell}, \tag{5.55}$$

where it is recalled that r_1 is the radius of the initial circular orbit. Then, the first impulse raises the apoapsis of the orbit such that the semi-major axis and eccentricity of the transfer orbit are given as

$$
\begin{aligned}
a &= \frac{r_1 + r_2}{2}, && (5.56)\\
e &= \frac{r_2 - r_1}{r_2 + r_1}. && (5.57)
\end{aligned}
$$

The magnitude Δv_1 of the first impulse $\Delta^{\mathcal{I}}\mathbf{v}_1$ is then given as

$$
\Delta v_1 = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} - \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} - v_1^-, \tag{5.58}
$$

where, because the initial orbit is circular with radius r_1, the circular speed on the initial orbit is given as

$$
v_1^- = \sqrt{\frac{\mu}{r_1}}. \tag{5.59}
$$

Now, the direction of the first impulse is equal to the direction of the inertial velocity at the point of application of $\Delta^{\mathcal{I}}\mathbf{v}_1$. Because the first impulse is applied tangentially at the position $\mathbf{r}_1$ and ${}^{\mathcal{I}}\mathbf{h}_1$ is orthogonal to $\mathbf{r}_1$, the unit vector in the direction of the inertial velocity at $\mathbf{r}_1$ on the first orbit, denoted $\mathbf{u}_1$, is given as

$$
\mathbf{u}_1 = \frac{{}^{\mathcal{I}}\mathbf{h}_1 \times \boldsymbol{\ell}}{\left\| {}^{\mathcal{I}}\mathbf{h}_1 \times \boldsymbol{\ell} \right\|} = \frac{{}^{\mathcal{I}}\mathbf{h}_1 \times \boldsymbol{\ell}}{\left\| {}^{\mathcal{I}}\mathbf{h}_1 \right\|}, \tag{5.60}
$$

where it is noted that $\mathbf{r}_1$ lies along the direction $\boldsymbol{\ell}$ given in Eq. (5.54) and $\left\| {}^{\mathcal{I}}\mathbf{h}_1 \times \boldsymbol{\ell} \right\| = \left\| {}^{\mathcal{I}}\mathbf{h}_1 \right\|$ because ${}^{\mathcal{I}}\mathbf{h}_1$ and $\boldsymbol{\ell}$ are orthogonal and $\boldsymbol{\ell}$ is a unit vector. The inertial velocity the instant before the first impulse is applied, denoted ${}^{\mathcal{I}}\mathbf{v}_1^-$, is then obtained as

$$
{}^{\mathcal{I}}\mathbf{v}_1^- = v_1^- \mathbf{u}_1, \tag{5.61}
$$

where v_1^- is obtained from Eq. (5.59). The first impulse of the transfer is then given as

$$
\Delta^{\mathcal{I}}\mathbf{v}_1 = \Delta v_1 \mathbf{u}_1, \tag{5.62}
$$

where Δv_1 is obtained from Eq. (5.58). Adding the results of Eqs. (5.61) and (5.62), the inertial velocity the instant after the application of the first impulse, denoted ${}^{\mathcal{I}}\mathbf{v}_1^+$, is given as

$$
{}^{\mathcal{I}}\mathbf{v}_1^+ = {}^{\mathcal{I}}\mathbf{v}_1^- + \Delta^{\mathcal{I}}\mathbf{v}_1. \tag{5.63}
$$

It is noted again that $\Delta^{\mathcal{I}}\mathbf{v}_1$ in Eq. (5.62) raises apoapsis because it is applied at the periapsis of the transfer orbit. Furthermore, the impulse $\Delta^{\mathcal{I}}\mathbf{v}_1$ does not change the orbital plane. Now, because the spacecraft is moving more slowly at apoapsis than at periapsis, the second impulse, $\Delta^{\mathcal{I}}\mathbf{v}_2$, performed at the apoapsis of the transfer orbit simultaneously changes the inclination and raises periapsis to the radius r_2 of the terminal orbit. The velocity of the spacecraft the instant before the first impulse is applied, denoted ${}^{\mathcal{I}}\mathbf{v}_2^-$, must lie in the same orbital plane as that of the initial inertial velocity, but in the direction opposite that of ${}^{\mathcal{I}}\mathbf{v}_1$. Furthermore, because the second impulse is applied at the apoapsis of the transfer orbit, the speed of the spacecraft the instant before the second impulse is applied, denoted v_2^-, is given as

$$v_2^- = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}}, \tag{5.64}$$

where the semi-major axis a in Eq. (5.64) is obtained from Eq. (5.56). Therefore, the inertial velocity the instant before the application of the second impulse, denoted ${}^{\mathcal{I}}\mathbf{v}_2^-$ is given as

$${}^{\mathcal{I}}\mathbf{v}_2^- = -v_2^- \mathbf{u}_1 \tag{5.65}$$

and v_2^- is obtained from Eq. (5.64). Note that, because the terminal orbit is circular with a radius r_2, the speed of the spacecraft on the terminal orbit, denoted v_2^+, is given as

$$v_2^+ = \sqrt{\frac{\mu}{r_2}}. \tag{5.66}$$

Note that, if the direction of the inertial velocity the instant after the second impulse can be determined, then the second impulse, $\Delta{}^{\mathcal{I}}\mathbf{v}_2$, can also be determined. It is observed that the inertial velocity the instant after the second impulse is applied must lie in the plane of the terminal orbit at the apoapsis of the transfer orbit. The position of the spacecraft at the apoapsis of the transfer orbit must have a magnitude r_2 (that is, the apoapsis of the transfer orbit is r_2) and must lie in the direction opposite $\boldsymbol{\ell}$. Therefore, the direction of the inertial velocity the instant after the second impulse is applied, denoted $\mathbf{u}_2$, is given as

$$\mathbf{u}_2 = \frac{{}^{\mathcal{I}}\mathbf{h}_2 \times (-\boldsymbol{\ell})}{\|{}^{\mathcal{I}}\mathbf{h}_2 \times (-\boldsymbol{\ell})\|} = -\frac{{}^{\mathcal{I}}\mathbf{h}_2 \times \boldsymbol{\ell}}{\|{}^{\mathcal{I}}\mathbf{h}_2\|}, \tag{5.67}$$

where $\|{}^{\mathcal{I}}\mathbf{h}_2 \times \boldsymbol{\ell}\| = \|{}^{\mathcal{I}}\mathbf{h}_2\|$ in Eq. (5.67) because ${}^{\mathcal{I}}\mathbf{h}_2$ and $\boldsymbol{\ell}$ are orthogonal and $\boldsymbol{\ell}$ is a unit vector. Therefore, the inertial velocity the instant after the second impulse is applied is given as

$${}^{\mathcal{I}}\mathbf{v}_2^+ = v_2^+ \mathbf{u}_2, \tag{5.68}$$

where v_2^+ is obtained from Eq. (5.66). The second impulse, $\Delta{}^{\mathcal{I}}\mathbf{v}_2$, is then given as

$$\Delta{}^{\mathcal{I}}\mathbf{v}_2 = {}^{\mathcal{I}}\mathbf{v}_2^+ - {}^{\mathcal{I}}\mathbf{v}_2^-. \tag{5.69}$$

The two velocities, ${}^{\mathcal{I}}\mathbf{v}_2^-$ and ${}^{\mathcal{I}}\mathbf{v}_2^+$, can then be used to determine the angle θ by which the velocity ${}^{\mathcal{I}}\mathbf{v}_2^-$ is rotated (where, from the result of Section 5.7.1, the angle θ is in general *not* equal to Δi, where Δi is the change in the orbital inclination). Specifically, the angle θ that defines the rotation of ${}^{\mathcal{I}}\mathbf{v}_2^-$ to attain ${}^{\mathcal{I}}\mathbf{v}_2^+$ is given as

$${}^{\mathcal{I}}\mathbf{v}_2^+ \cdot {}^{\mathcal{I}}\mathbf{v}_2^- = v_2^+ v_2^- \cos\theta \tag{5.70}$$

which implies that

$$\theta = \cos^{-1}\left(\frac{{}^{\mathcal{I}}\mathbf{v}_2^+ \cdot {}^{\mathcal{I}}\mathbf{v}_2^-}{v_2^+ v_2^-}\right). \tag{5.71}$$

Alternatively, the angle θ can be computed using the initial and terminal specific angular momenta as

$${}^{\mathcal{I}}\mathbf{h}_1 \cdot {}^{\mathcal{I}}\mathbf{h}_2 = \left\|{}^{\mathcal{I}}\mathbf{h}_1\right\| \left\|{}^{\mathcal{I}}\mathbf{h}_2\right\| = h_1 h_2 \cos\theta \tag{5.72}$$

which implies that

$$\theta = \cos^{-1}\left(\frac{{}^{\mathcal{I}}\mathbf{h}_1 \cdot {}^{\mathcal{I}}\mathbf{h}_2}{h_1 h_2}\right). \tag{5.73}$$

Again, it is re-emphasized that $\theta \neq \Delta i$. Finally, it is noted for completeness that the magnitude of $\Delta^{\mathcal{I}}\mathbf{v}_2$ in Eq. (5.69) is given as

$$(\Delta v_2)^2 = (v_2^-)^2 + (v_2^+)^2 - 2v_2^- v_2^+ \cos\theta \tag{5.74}$$

or, equivalently, as

$$\Delta v_2 = \left\|\Delta^{\mathcal{I}}\mathbf{v}_2\right\| = \left\|{}^{\mathcal{I}}\mathbf{v}_2^+ - {}^{\mathcal{I}}\mathbf{v}_2^-\right\|. \tag{5.75}$$

A schematic of the two-impulse non-co-planar transfer between two circular orbits of radii i_1 and i_2 is shown in Fig. 5.8.

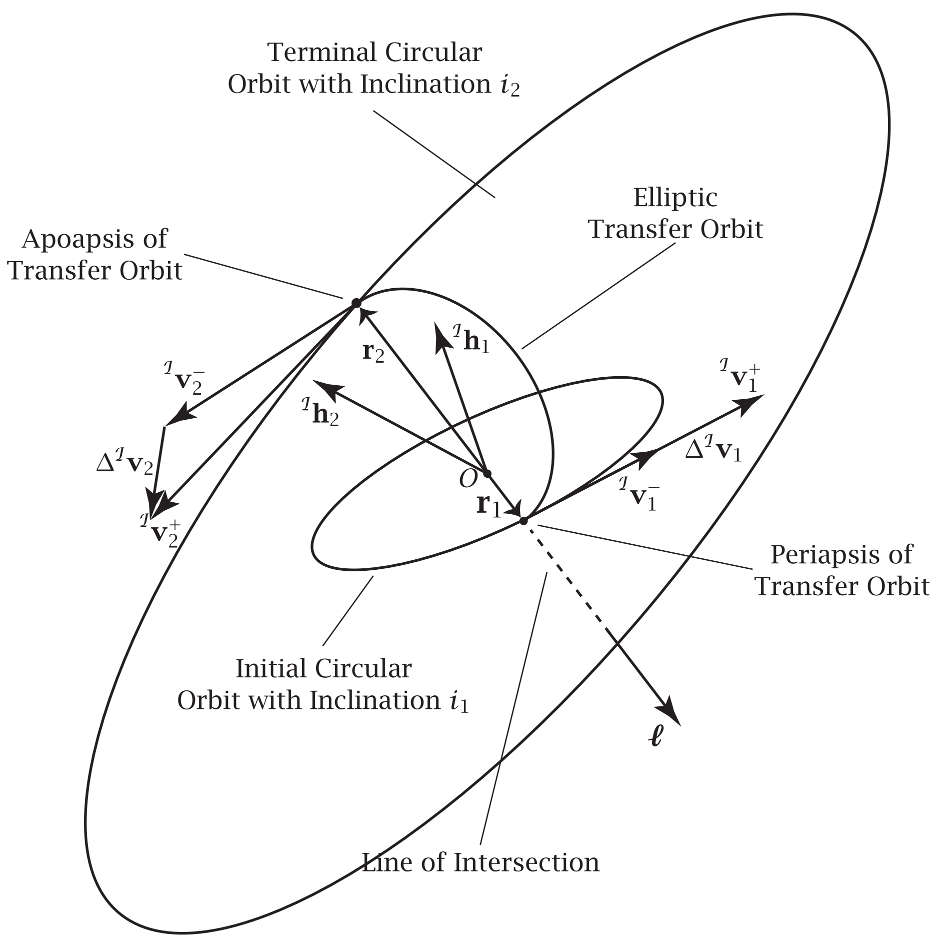


Figure 5.8 Schematic of two-impulse non-co-planar transfer between two circular orbits with different inclinations.

Problems for Chapter 5

5–1 Suppose that it is desired to transfer a spacecraft from an initial circular orbit of radius r_1 to a terminal co-planar circular orbit of radius r_2. Suppose that it is desired to use either a Hohmann transfer, a bi-elliptic transfer, or a bi-parabolic transfer. Determine the following information to decide which transfer to use.

(a) For a Hohmann transfer, determine expressions for the magnitude of the two impulses, Δv_1 and Δv_2. Nondimensionalize the two impulses by determining the ratios $\Delta v_1/v_{c1}$ and $\Delta v_2/v_{c1}$ and as functions of the quantity $R = r_2/r_1$.

(b) For a bi-elliptic transfer, determine expressions for the magnitude of the three impulses, Δv_1, Δv_2, and Δv_3. Nondimensionalize the three impulses by determining the ratios $\Delta v_1/v_{c1}$, $\Delta v_2/v_{c1}$, and $\Delta v_3/v_{c1}$ and as functions of the quantities $R = r_2/r_1$ and $S = r_i/r_2$ (where r_i is the apoapsis of the intermediate transfer orbit used in the bi-elliptic transfer).

(c) For a bi-parabolic transfer, determine expressions for the magnitude of the two impulses, Δv_1 and Δv_2. Nondimensionalize the two impulses by determining the ratios $\Delta v_1/v_{c1}$ and $\Delta v_2/v_{c1}$ and as functions of the quantity $R = r_2/r_1$.

(d) Make the following two plots in MATLAB of the normalized impulse, $\Delta v/v_{c1}$, for each transfer as a function of R, where R is the "x"–axis and $\Delta v/v_{c1}$ is the "y"–axis. For use $R \in [1, 20]$ and do not change the default settings for the "y"–axis in MATLAB. For the second plot, change the range for R to be such that $R \in [10, 16]$ and change the range for $\Delta v/v_{c1}$ to be $\Delta v/v_{c1} \in [0.51, 0.55]$. When making both plots, use the values $S = (2, 5, 10, 11, 12, 15)$ for the bi-elliptic transfer. For each plot place all of the lines on the same plot (that is, put the Hohmann transfer, all bi-elliptic transfers, and the bi-parabolic transfer on the same plot).

5–2 Suppose now it is desired to determine the values of the ratio $R = r_2/r_1$ that determines crossover points where the Hohmann transfer becomes less economical that either a bi-elliptic transfer or the bi-parabolic transfer. Using the results of Question 1, solve the following root-finding problems using either the MATLAB root-finder `fsolve` or your own root-finder:

(a) the value of R where the total impulse of the Hohmann transfer is the same as the total impulse of the bi-parabolic transfer;

(b) the values of R where the total impulse of the Hohmann transfer is the same as the total impulse of the bi-elliptic transfers for $S = r_i/r_2 = (2, 5, 10, 11, 12, 15)$ (where r_i is the apoapsis of the intermediate transfer orbit used in the bi-elliptic transfer as given in Question 1);

(c) Plot the roots obtained in part (b) as a function of S.

5–3 A spacecraft is in a circular orbit that has a speed of unity in canonical units (that is, $\mu = 1$). From this starting orbit the goal is to rendezvous with a spacecraft that in another co-planar circular orbit with a speed of 0.5 canonical units. Determined which of the Hohmann, bi-elliptic, or bi-parabolic transfer accomplishes the orbit transfer with the lowest impulse. Using MATLAB, plot the initial orbit, the terminal orbit, and the transfer orbit on the same two-dimensional plot. Include the impulses required to accomplish the orbit transfer on your plot using the MATLAB command `quiver`.

5–4 Consider a spacecraft in a circular low-Earth orbit with altitude 300 km, inclination 57 deg, and longitude of the ascending node 60 deg. Suppose that the objective is to transfer the spacecraft to a geostationary earth orbit using a Hohmann transfer. Knowing that a geostationary orbit is a circular orbit with a period of 23.934 hours based on a sidereal day (as opposed to a solar day), determine the following quantities:

(a) The magnitude of each impulse that contributes to the total Δv (in $\text{km} \cdot \text{s}^{-1}$) required to complete the transfer, where the inclination change is performed at the apoapsis of the transfer orbit (that is, the second impulse both circularizes the final orbit and accomplishes the inclination change);

(b) The total ΔV required to complete the transfer;

(c) The time (in hours) required to complete the transfer;

(d) Assuming that the rocket engine has a specific impulse of 320 s, determine the ratio of the initial and terminal masses due to the magnitude of each impulse obtained in part (a).

(e) Using MATLAB, plot on the same three-dimensional plot the initial orbit, the terminal orbit, the transfer orbit, and the line of intersection between the initial and terminal orbits. Include the impulses required to accomplish the orbit transfer on your plot using the MATLAB command quiver3.

(f) Does changing the longitude of the ascending node change the location on the initial orbit where the transfer starts?

5–5 A Global Positioning System (GPS) spacecraft is launched from the Eastern Test Range (ETR) at the Cape Canaveral Air Force Station and initially inserted into a circular low-Earth orbit (LEO) at an altitude of 350 km with an orbital inclination of 28 deg. From this initial orbit the goal is to transfer the spacecraft to a final circular GPS orbit of radius 26558 km with an orbital inclination of 55 deg using a two-impulse transfer such that the impulses are applied along the line of intersection between the two orbits. Determine the following information:

(a) The line of intersection in Earth-centered inertial (ECI) coordinates.

(b) The positions of the spacecraft $\mathbf{r}_1$ and $\mathbf{r}_2$ that define the locations where the two impulses $\Delta^{\mathcal{I}}\mathbf{v}_1$ and $\Delta^{\mathcal{I}}\mathbf{v}_2$ are applied.

(c) The total Δv (in $\text{km} \cdot \text{s}^{-1}$) if the required inclination change is performed purely at the apoapsis of the transfer orbit (that is, the second impulse both circularizes the final orbit and accomplishes the inclination change).

(d) The time of flight (in hours) of the transfer ellipse.

(e) The eccentricity of the transfer orbit.

(f) The angle θ that defines the rotation of the orbital plane due to the application of the second impulse.

(g) Using MATLAB, plot on the same three-dimensional plot the initial orbit, the terminal orbit, the transfer orbit, and the line of intersection between the initial and terminal orbits. Include the impulses required to accomplish the orbit transfer on your plot using the MATLAB command quiver3.

Assume in your answers that the longitude of the ascending node of the initial and terminal orbits is the same.

5–6 A spacecraft is launched from the Kennedy Space Center in Florida into an initial circular low-Earth orbit (LEO) with altitude 300 km and an inclination $i = 28.5$ deg. The goal is to transfer the spacecraft to a geostationary orbit (GEO), where it is noted that a geostationary orbit is an circular equatorial orbit with a period of 24 hours). Suppose that it is desired to transfer the spacecraft from the given LEO to GEO using a two-impulse transfer that consists of two energy change impulses along with up to two inclination change impulses. Suppose further that f and $1 - f$ denote, respectively, the fraction of the inclination change that is accomplished at the initial LEO and the apoapsis of the transfer orbit (that is, the inclination change is divided into an inclination change that is accomplished at the radius of the initial LEO while the remainder of the inclination change is accomplished at the apoapsis of the transfer orbit). Using the information provided, determine the following:

(a) The magnitude of the total impulse assuming that all of the inclination change is accomplished at the apoapsis of the elliptic transfer orbit;

(b) The magnitude of the total impulse assuming that all of the inclination change is accomplished at the initial LEO;

(c) A two-dimensional plot in MATLAB that shows the total impulse normalized by the initial circular speed (that is $\Delta v / v_{c1}$) as a function of the fraction f of the total inclination change that is accomplished at the initial LEO.

(d) From the plot generated in part (c) determine the value of f that results in the smallest total impulse for the maneuver.

In obtaining the plot in part (c) above, use increments of 0.01 for f. In other words, solve for the total impulse in increments of 0.01 for $f \in [0, 1]$.

Chapter 6

Interplanetary Orbital Transfer

6.1 Introduction

An important aspect of orbital transfer arises when a spacecraft is transferred from an orbit about an initial planet and inserted into an orbit about a terminal planet. Any orbital transfer where the centrally attracting planet about which it is orbiting is referred to as an *interplanetary transfer*. In this chapter interplanetary transfers are considered using a so called *patched conic method*. The patched conic method relies on an approximation known as the *sphere of influence*, where the sphere of influence is the locus of points such that the force of gravitational attraction of a particular planet on the spacecraft dominates the force of gravitational attraction of the Sun on the spacecraft. The patched conic method enables using a series of two-body approximations where the spacecraft is considered to be under the gravitational influence of either a departure or arrival planet when the spacecraft lies within the sphere of influence of that planet and is under the gravitational influence of only the Sun when the spacecraft lies outside of the sphere of influence of any planet. First, an interplanetary Hohmann transfer between two planets is developed. Second, rendezvous opportunities that provide the appropriate timing with an arrival planet from a departure planet are considered using the interplanetary Hohmann transfer as the basis of such a rendezvous opportunity. Third, the sphere of influence of a planet is derived assuming that the spacecraft is under the influence of both a planet that orbits the Sun and the Sun itself. Fourth, the patched conic approximation method is developed. The patched conic method is divided into two parts: planetary departure and planetary arrival. In planetary departure, the conditions are developed that make it possible for the spacecraft to leave the sphere of influence of a departure planet. Similarly, in planetary arrival, the conditions are developed that enable a spacecraft to enter the sphere of influence of an arrival planet. The patched conic method is developed for the case where a spacecraft departs from an inner planet and arrives at an outer planet and the case where a spacecraft departs from an outer planet and arrives at an inner planet. In the case of planetary departure the spacecraft must leave the sphere of influence of the departure planet along a hyperbolic trajectory relative to the departure planet. Similarly, in the case of planetary arrival the spacecraft must enter the sphere of influence of the arrival planet along a hyperbolic trajectory relative to the departure planet. Because the planet is moving along a hyperbolic trajectory relative to an arrival planet, two options are possible. The first option is that a capture impulse can be applied in order to place the spacecraft into an orbit relative to the arrival planet. The second options is that the spacecraft can perform a planetary flyby, also known as a *gravity assist*, that will enable the velocity of the spacecraft relative to the Sun be changed without the need to consume any fuel (that is, the impulse due to the planetary flyby is obtained without propulsion). The planetary flyby can either send the spacecraft further from the Sun or closer to the Sun depending upon whether the intent of the gravity assist is to transfer the spacecraft to an outer planet or an inner planet.

6.2 Interplanetary Hohmann Transfer

An interesting feature of most of the planets in the solar system is that they lie in nearly the same orbital plane as the ecliptic plane (that is, the plane that contains the orbit of the Earth). The greatest deviations in inclination from the inclination of the orbit of the Earth are found in the innermost planet, Mercury, which has an inclination of 7 deg, and the dwarf planet, Pluto, which has an inclination of 17 deg. All of the other planets have inclinations that are within 3.5 deg of the ecliptic plane. Because the orbital inclinations of the planets are close to the orbital inclination of the Earth, it will be assumed for simplicity in all of the derivations that follow that all planets lie in the same orbital plane.

As a starting point for interplanetary transfers, consider two planets in co-planar circular orbits relative to the Sun. The radius of the orbit of the first planet, which will be denoted the *departure planet*, is R_1, while the radius of the orbit of the second planet, which will be denoted the *arrival planet*, is R_2. Suppose further $R_2 > R_1$, that is, the departure planet is, relatively speaking, the *inner planet* while the arrival planet is the *outer planet*). Furthermore, let the gravitational parameter of the Sun be denoted μ_s. Taking the Sun to be an inertial reference frame, the circular speed of the departure planet relative to the Sun is given as

$$V_{c1} = \sqrt{\frac{\mu_s}{R_1}}. \tag{6.1}$$

Suppose now that a spacecraft has been placed onto a heliocentric transfer orbit from the departure planet such that the semi-major axis of the transfer orbit is $a = (R_1 + R_2)/2$. It is seen that the periapsis of this heliocentric orbit, where the periapsis is of the heliocentric orbit is denoted *perihelion*, the speed of the spacecraft on this heliocentric transfer orbit is given as

$$V_p = \sqrt{\frac{2\mu_s}{R_1} - \frac{\mu_s}{a}} = \sqrt{\frac{2\mu_s}{R_1} - \frac{2\mu_s}{R_1 + R_2}} = \sqrt{\frac{\mu_s}{R_1}} = \sqrt{\frac{\mu_s}{R_1}}\sqrt{\frac{2R_2}{R_1 + R_2}} \tag{6.2}$$

The departure impulse, applied at perihelion, that raises the apohelion of from R_1 to a, thus placing the spacecraft onto the aforementioned heliocentric transfer orbit, is the difference between V_p given in Eq. (6.1) and V_{c1} given in Eq. (6.2), that is,

$$\Delta V_D = V_p - V_{c1} = \sqrt{\frac{\mu_s}{R_1}}\sqrt{\frac{2R_2}{R_1 + R_2}} - \sqrt{\frac{\mu_s}{R_1}} = \sqrt{\frac{\mu_s}{R_1}}\left(\sqrt{\frac{2R_2}{R_1 + R_2}} - 1\right). \tag{6.3}$$

Next, the circular speed of the arrival planet relative to the Sun is given as

$$V_{c2} = \sqrt{\frac{\mu_s}{R_2}}. \tag{6.4}$$

Furthermore, the speed of the spacecraft at the apohelion of the transfer orbit is given as

$$V_a = \sqrt{\frac{2\mu_s}{R_2} - \frac{\mu_s}{a}} = \sqrt{\frac{2\mu_s}{R_2} - \frac{2\mu_s}{R_1 + R_2}} = \sqrt{\frac{\mu_s}{R_2}}\sqrt{\frac{2R_1}{R_1 + R_2}}. \tag{6.5}$$

Then, the impulse required in order to place the spacecraft into a circular heliocentric orbit of radius R_2 is the difference between the apohelion speed, V_a, given in Eq. (6.5) and the circular speed of the arrival planet, V_{c2}, that is,

$$\Delta V_A = \sqrt{\frac{\mu_s}{R_2}} - \sqrt{\frac{\mu_s}{R_2}}\sqrt{\frac{2R_1}{R_1 + R_2}} = \sqrt{\frac{\mu_s}{R_2}}\left(1 - \sqrt{\frac{2R_1}{R_1 + R_2}}\right). \tag{6.6}$$

It is noted that if the departure planet is the outer planet while the arrival planet is the inner planet then the impulses ΔV_D and ΔV_A are in the opposite directions from those given Eqs. (6.3) and (6.6). Finally, it is important to note that an interplanetary Hohmann transfer must be timed appropriately in order to leave the departure planet at a time such that the arrival planet it is appropriately located when the spacecraft reaches the apohelion of the transfer orbit. An appropriate chosen departure time leads to a *rendezvous opportunity* where, upon application of the second impulse ΔV_2 given in Eq. (6.6), the spacecraft and the arrival planet have not only the same circular speed, $V_{c2} = \sqrt{\mu_s/R_2}$, but are also in the same location.

6.3 Rendezvous of a Spacecraft with a Planet

Suppose now that it is desired to perform a orbital transfer of a spacecraft from an orbit that is equivalent to the orbit of a departure planet to an orbit that is equivalent to the orbit of an arrival planet. Such a transfer falls into the category of an interplanetary Hohmann transfer as given in Section 6.2. The difference, however, between a standard Hohmann transfer between two circular orbits and a Hohmann transfer that departs from one planet and arrive at another planet lies in the fact the latter such transfer must be timed such that the spacecraft leaves the orbit of the departure planet at a time such that the arrival planet and the spacecraft are co-located at the time of arrival. Furthermore, because upon arrival the spacecraft must be moving with the same velocity as that of the arrival planet, it is necessary that the spacecraft and the arrival planet achieve a rendezvous upon arrival. Thus, transferring a spacecraft from an departure planet to an arrival planet requires that such a rendezvous opportunity occur.

In order realize a rendezvous opportunity, consider the following configuration of two planets, denoted planet 1 and planet 2, and a spacecraft. Because the planets move in co-planar circular orbits, an arbitrary line of apsides can be used where this line of apsides contains the perihelion and the apohelion of the transfer orbit. Using this line of apsides, the true anomalies of the orbits of planet 1 and planet 2 are given, respectively, as

$$\nu_1 = \nu_{10} + n_1 t, \tag{6.7}$$

$$\nu_2 = \nu_{20} + n_2 t, \tag{6.8}$$

where

$$n_1 = \frac{2\pi}{\tau_1}, \tag{6.9}$$

$$n_2 = \frac{2\pi}{\tau_2} \tag{6.10}$$

are the mean motions of planet 1 and planet 2, respectively (where the mean motion is defined as the angular velocity of the line that connects the Sun to each of the planets and τ_1 and τ_2 are the orbital periods of planet 1 and planet 2, respectively) and ν_{10} and ν_{20} are the true anomalies of planet 1 and planet 2, respectively, at time $t = 0$. Given that the orbits of each planet are assumed to be circular, the angles ν_1 and ν_2 can be arbitrarily measured from a common line of apsides relative to the Sun. Assume that this common line of apsides is the one associated with the interplanetary elliptic Hohmann transfer orbit. The phase angle between the two planets, denoted ϕ, is then given as

$$\phi = \nu_2 - \nu_1. \tag{6.11}$$

Substituting the results of Eqs. (6.7) and (6.8) into Eq. (6.11) gives

$$\phi = \nu_{20} - \nu_{10} + (n_2 - n_1)t = \phi_0 + (n_2 - n_1)t, \tag{6.12}$$

where

$$\phi_0 = \nu_{20} - \nu_{10} \tag{6.13}$$

is the phase angle at time $t = 0$. Consider now the following two possibilities: $n_1 > n_2$ and $n_1 < n_2$. First, for the case $n_1 > n_2$ we have

$$\frac{2\pi}{\tau_1} > \frac{2\pi}{\tau_2} \tag{6.14}$$

which implies that

$$\tau_1 < \tau_2. \tag{6.15}$$

As a result, the orbital period of planet 1 is less than the orbital period of planet 2 which implies that orbit 1 is *smaller* than orbit 2. Next, for the case $n_1 < n_2$ we have

$$\frac{2\pi}{\tau_1} > \frac{2\pi}{\tau_2} \tag{6.16}$$

which implies that

$$\tau_1 > \tau_2. \tag{6.17}$$

As a result, the orbital period of planet 1 is greater than the orbital period of planet 2 which implies that orbit 1 is *larger* than orbit 2.

Suppose now that we consider a time period τ_s where the phase angle ϕ either increases or decreases by one full rotation, that is, ϕ either increases or decreases by 2π. The time period for a one period change in ϕ is called the *synodic period* between the two planets. The synodic period is now computed for both cases $n_1 < n_2$ and $n_1 > n_2$. First, for the case $n_1 < n_2$, planet 1 is moving with a larger speed than planet 2. As a result, the angle ν_2 changes more slowly than the angle ν_1 and the phase angle given in Eq. (6.11) *decreases* as time increases. Consequently, the synodic period τ_s will be the time for ϕ to have *decreased* by 2π. Then, from Eq. (6.12) we have

$$\phi_0 - 2\pi = \phi_0 + (n_2 - n_1)\tau_s. \tag{6.18}$$

Solving Eq. (6.18) for the synodic period τ_s gives

$$\tau_s = \frac{2\pi}{n_1 - n_2}, \quad (n_1 > n_2). \tag{6.19}$$

Next, for the case $n_1 < n_2$, planet 1 is moving with a smaller speed than planet 2. As a result, the angle ν_2 changes more rapidly than the angle ν_1 and the phase angle given in Eq. (6.11) *increases* as time increases. Consequently, the synodic period τ_s will be the time for ϕ to have *increased* by 2π. Then, from Eq. (6.12) we have

$$\phi_0 + 2\pi = \phi_0 + (n_2 - n_1)\tau_s. \tag{6.20}$$

Solving Eq. (6.20) for the synodic period τ_s gives

$$\tau_s = \frac{2\pi}{n_2 - n_1}, \quad (n_1 < n_2). \tag{6.21}$$

Combining the results of Eqs. (6.19) and (6.21), the synodic period is given more generally as

$$\tau_s = \frac{2\pi}{|n_2 - n_1|}. \tag{6.22}$$

Finally, noting that $n_1 = 2\pi/\tau_1$ and $n_2 = 2\pi/\tau_2$, the synodic period can be written in terms of the periods of the two planets as

$$\tau_s = \frac{2\pi}{\left|\frac{2\pi}{\tau_2} - \frac{2\pi}{\tau_1}\right|} = \frac{\tau_2\tau_1}{|\tau_2 - \tau_1|}. \tag{6.23}$$

Because the synodic period defines the time when the phase angle has changed by 2π, the synodic period defines the orbital period of the motion of planet 2 relative to the motion of planet 1.

Consider now an interplanetary Hohmann transfer between planet 1 and planet 2. The time required to complete the transfer is the half-period of the elliptic transfer orbit, that is,

$$t_T = \pi\sqrt{\frac{a^3}{\mu_s}}. \tag{6.24}$$

Noting that the semi-major axis of the transfer orbit is $a = (R_1 + R_2)/2$ (where R_1 and R_2 are the distances from the Sun to planet 1 and planet 2, respectively), Eq. (6.24) can be written as

$$t_T = \pi\sqrt{\frac{(R_1 + R_2)^3}{8\mu_s}}. \tag{6.25}$$

Now, it is seen that the change in the true anomaly on the transfer orbit from planet 1 to planet 2 is π (that is, the spacecraft starts at the perihelion of the transfer orbit and terminates at the apohelion of the transfer orbit). Furthermore, because the spacecraft arrives at the apohelion of the transfer orbit and must meet planet 2 at this point, upon arrival of the spacecraft at planet 2 it must be the case that planet 2 is located on exactly the opposite side from where planet 1 was located when the spacecraft departed planet 1. Thus, in the time t_T give in Eq. (6.25) the location of planet 2 must have changed by an angle $n_2 t_T$, where n_2 is the angular velocity of the direction from the Sun to planet 2. Moreover, denoting the initial phase angle between planet 1 and planet 2 by ϕ_0, it must be the case that

$$\phi_0 + n_2 t_T = \pi \tag{6.26}$$

Solving Eq. (6.26) for ϕ_0 gives

$$\phi_0 = \pi - n_2 t_T. \tag{6.27}$$

The final phase angle (which is the phase angle when the spacecraft arrives at planet 2), denoted ϕ_f, is given from Eq. (6.12) as

$$\phi_f = \phi_0 + (n_2 - n_1)t_T = \pi - n_2 t_T + (n_2 - n_1)t_T = \pi - n_1 t_T. \tag{6.28}$$

Consider now a roundtrip of a spacecraft between planet 1 and planet 2 using an interplanetary Hohmann transfer on both legs of the transfer. First, the spacecraft is transferred from planet 1 to planet 2 using an interplanetary Hohmann transfer that raises both apohelion and perihelion (in that order). Then, when possible, the spacecraft is returned to planet 1 from planet 2 using an interplanetary Hohmann transfer that lowers both perihelion and apohelion (in that order). In order to accomplish both the outbound and inbound transfers, it is necessary that the spacecraft rendezvous with planet 2 on the outbound transfer while the spacecraft must rendezvous with planet 1 on the inbound transfer. The outbound and inbound transfers are shown in Figs. 6.1(a) and 6.1(b), respectively. Now because both the outbound and inbound transfers are Hohmann transfers, the outbound transfer orbit (that is, the transfer orbit from planet 1 to planet 2) must be the first half (that is, from perihelion to apohelion) of an elliptic transfer orbit corresponding to a Hohmann transfer while the return transfer orbit (that is, the transfer orbit from planet 2 to planet 1) must be the second half (that is, from apohelion to perihelion) of an elliptic transfer orbit corresponding to a Hohmann transfer. Therefore, the duration of the segment on both the outbound transfer orbit and the inbound transfer obit must be the same. Suppose that the time taken to accomplish either the outbound or the inbound transfer is denoted t_T as given in Eq. (6.24). Because the durations of both the outbound and the inbound transfers are the same, the change in true anomaly of planet 1 during the outbound trip must be the same as the change in true anomaly of planet 1 during the return trip. Consequently, the phase angle at the start of the return transfer from planet 2 to planet 1 must be exactly opposite the phase angle at the end of the outbound transfer from planet 1 to planet 2, that is,

$$\phi_0' = -\phi_f, \tag{6.29}$$

where ϕ_0' is the phase angle at the start of the return transfer from planet 2 to planet 1 and ϕ_f is obtained from Eq. (6.28). Suppose now that the time is reset to zero upon the arrival of the spacecraft at planet 2. Then, because $\phi = \phi_f$ upon arrival at planet 2 on the outbound transfer, the phase angle measured from the time of arrival at planet 2 is obtained from Eq. (6.12) as

$$\phi' = \phi_f + (n_2 - n_1)t. \tag{6.30}$$

Suppose now that the quantity t_w denotes the time required to wait before a return transfer can be started. Noting that the phase angle at the start of the return transfer, that is at time t_w, must be $\phi'(t_w) = -\phi_f$, the wait time t_w is obtained from Eq. (6.30) as

$$\phi'(t_w) = -\phi_f = \phi_f + (n_2 - n_1)t_w. \tag{6.31}$$

Solving Eq. (6.31) for t_w gives

$$t_w = -\frac{2\phi_f}{n_2 - n_1}, \tag{6.32}$$

where it is noted again that ϕ_f is as obtained in Eq. (6.28). It can be seen that the result of Eq. (6.32) could be either positive or negative. If t_w is negative, then the opportunity to return from planet 2 to planet 1 occurred in the past. Thus, the next opportunity to accomplish a return transfer from planet 2 to planet 1 will occur every τ_s time units (that is, every synodic period). Recall for the case $n_1 > n_2$ that the phase angle decreases. Therefore, using the results of Eqs. (6.19) and (6.32), the wait times are given as

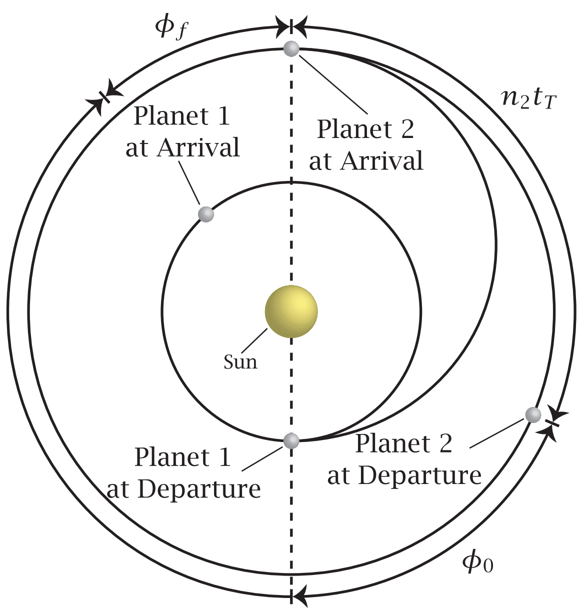
$$t_w = -\frac{2\phi_f}{n_2 - n_1} + k\tau_s = -\frac{2\phi_f}{n_2 - n_1} + k\frac{2\pi}{n_1 - n_2} = -\frac{2\phi_f + 2\pi k}{n_2 - n_1}. \tag{6.33}$$

Next, recall for the case $n_1 < n_2$ that the phase angle increases. Therefore, using the results of and (6.21) and (6.32), the wait times are given as

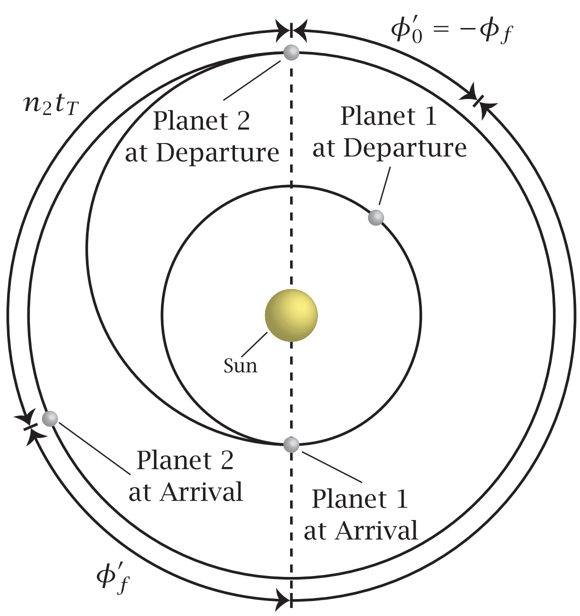
$$t_w = -\frac{2\phi_f}{n_2 - n_1} + k\tau_s = -\frac{2\phi_f}{n_2 - n_1} + k\frac{2\pi}{n_2 - n_1} = -\frac{2\phi_f - 2\pi k}{n_2 - n_1}. \tag{6.34}$$

Combining the results of Eqs. (6.33) and (6.34), the possible wait times to accomplish a return transfer from planet 2 to planet 1 are given as

$$t_w = \begin{cases} -\dfrac{2\phi_f + 2\pi k}{n_2 - n_1} & , \quad n_1 > n_2, \\ -\dfrac{2\phi_f - 2\pi k}{n_2 - n_1} & , \quad n_1 < n_2. \end{cases} \qquad k = 1, 2, \ldots \tag{6.35}$$



(a) Outbound Hohmann transfer from planet 1 to planet 2.



(b) Inbound Hohmann transfer from planet 2 to planet 1.

Figure 6.1 Schematic of outbound and inbound interplanetary Hohmann transfers between two planets.

6.4 Sphere of Influence

Suppose now that we consider the effect of the gravitational field of both the Sun and a planet on a spacecraft. The Sun is the body in the solar system that has the greatest gravitational influence. In fact, the Sun has a mass that is over three orders of magnitude larger than the most massive planet in the solar system (that planet being Jupiter) and is over five orders of magnitude more massive than the Earth. While the Sun is so massive and exerts a large gravitational force on any body, when a spacecraft is in the vicinity of a planet (that is, the spacecraft is within some distance of the planet) the gravitational force exerted by the planet on the spacecraft is larger than the gravitational force exerted by the Sun. The goal of this section is to provide a derivation of the classical *sphere of influence* that provides an approximation of the distance within which a planet has a larger gravitational effect on a spacecraft in comparison to that of the Sun. In the context of interplanetary trajectories, the sphere of influence provides a distance between the spacecraft and a planet where a transition takes place between the Sun being considered as the centrally attracting body and the planet being considered as the centrally attracting body.

In order to determine the sphere of influence of a planet, consider the three-body system of the Sun, denoted s, a planet, denoted p, and the spacecraft, denoted v, as shown in Fig. 6.2. The mass of the Sun, planet, and spacecraft are given as m_s, m_p,

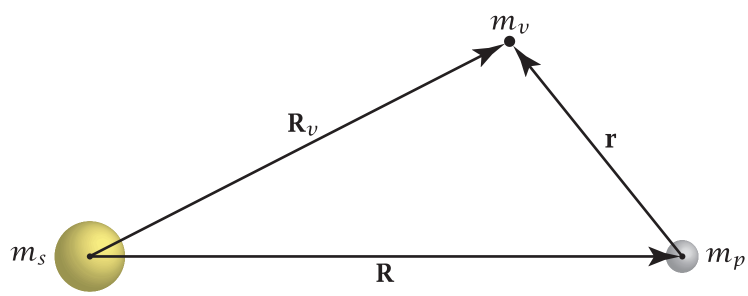


Figure 6.2 Three-body system consisting of the Sun, a planet, and a spacecraft.

and m_v, respectively. In addition, the position of the spacecraft relative to the Sun, the planet relative to the Sun, and the spacecraft relative to the planet are denoted, respectively, as $\mathbf{R}_v$, $\mathbf{R}$, and $\mathbf{r}$. Furthermore, denote the magnitudes of these three positions as

$$\|\mathbf{R}_v\| = R_v, \tag{6.36}$$

$$\|\mathbf{R}\| = R, \tag{6.37}$$

$$\|\mathbf{r}\| = r. \tag{6.38}$$

The position of the spacecraft relative to the Sun is then given as

$$\mathbf{R}_v = \mathbf{R} + \mathbf{r}. \tag{6.39}$$

Assume now that the only forces acting on any one of the three bodies are the gravitational forces of attraction of the other body. Using the terminology of Chapter 1 as given in Eq. (1.2) on page 9 (where $\mathbf{F}_{AB}$ was denoted the force exerted by object B on object A), the forces of gravitational attraction exerted by the Sun on the spacecraft, the

Sun on the planet, and the planet on the spacecraft are given from Newton's universal law of gravitation, respectively, as

$$\mathbf{F}_{vs} = -\frac{Gm_v m_s}{R_v^3}\mathbf{R}_v, \tag{6.40}$$

$$\mathbf{F}_{ps} = -\frac{Gm_p m_s}{R^3}\mathbf{R}, \tag{6.41}$$

$$\mathbf{F}_{vp} = -\frac{Gm_v m_p}{r^3}\mathbf{r}. \tag{6.42}$$

Suppose now that the Sun is considered the inertial reference frame, denoted $\mathcal{I}$, for the three-body system under consideration. Then the inertial acceleration of the spacecraft and the planet are given, respectively, as ${}^{\mathcal{I}}\mathbf{a}_v$ and ${}^{\mathcal{I}}\mathbf{a}_p$. Applying Newton's second law to the spacecraft gives

$$\mathbf{F}_v = \mathbf{F}_{vs} + \mathbf{F}_{vp} = -\frac{Gm_v m_s}{R_v^3}\mathbf{R}_v - \frac{Gm_v m_p}{r^3}\mathbf{r} = m_v {}^{\mathcal{I}}\mathbf{a}_v. \tag{6.43}$$

The inertial acceleration of the spacecraft is then given as

$${}^{\mathcal{I}}\mathbf{a}_v = -\frac{Gm_s}{R_v^3}\mathbf{R}_v - \frac{Gm_p}{r^3}\mathbf{r} = \mathbf{A}_s + \mathbf{P}_p, \tag{6.44}$$

where the quantities $\mathbf{A}_s$ and $\mathbf{P}_p$ are defined as

$$\mathbf{A}_s = -\frac{Gm_s}{R_v^3}\mathbf{R}_v, \tag{6.45}$$

$$\mathbf{P}_p = -\frac{Gm_p}{r^3}\mathbf{r}, \tag{6.46}$$

It is noted that $\mathbf{A}_s$ and $\mathbf{P}_p$ represent the gravitational accelerations due to the Sun and the planet on the vehicle, where $\mathbf{A}_s$ is considered the primary gravitational acceleration while $\mathbf{P}_p$ is considered the secondary acceleration or, equivalently, the perturbation from central-body Sun gravitation on the spacecraft. The magnitudes of $\mathbf{A}_s$ and $\mathbf{P}_p$ are then given as

$$A_s = \|\mathbf{A}_s\| = \frac{Gm_s}{R_v^2}, \tag{6.47}$$

$$P_p = \|\mathbf{P}_p\| = \frac{Gm_p}{r^2}. \tag{6.48}$$

Noting that $R_v \approx R$ (that is, the distance from the Sun to the spacecraft is approximately the same as the distance of the Sun to the planet), the quantity A_s in Eq. (6.47) can be approximated as

$$A_s = \frac{Gm_s}{R^2}. \tag{6.49}$$

Next, applying Newton's second law to the planet gives

$$\mathbf{F}_p = \mathbf{F}_{ps} + \mathbf{F}_{pv} = -\frac{Gm_p m_s}{R^3}\mathbf{R} + \frac{Gm_p m_v}{r^3}\mathbf{r} = m_p {}^{\mathcal{I}}\mathbf{a}_p. \tag{6.50}$$

where from Newton's third law it is noted that $\mathbf{F}_{pv} = -\mathbf{F}_{vp}$. The inertial acceleration of the spacecraft is then given as

$${}^{\mathcal{I}}\mathbf{a}_p = -\frac{Gm_s}{R^3}\mathbf{R} + \frac{Gm_v}{r^3}\mathbf{r} = \frac{Gm_v}{r^3}\mathbf{r} - \frac{Gm_s}{R^3}\mathbf{R}. \tag{6.51}$$

The inertial acceleration of the spacecraft relative to the planet is then given as

$$\begin{aligned}{}^{\mathcal{I}}\mathbf{a}_v - {}^{\mathcal{I}}\mathbf{a}_p = {}^{\mathcal{I}}\mathbf{a}_{v/p} &= -\frac{Gm_s}{R_v^3}\mathbf{R}_v - \frac{Gm_p}{r^3}\mathbf{r} - \left[\frac{Gm_v}{r^3}\mathbf{r} - \frac{Gm_s}{R^3}\mathbf{R}\right] \\ &= -\frac{G(m_p + m_v)}{r^3}\mathbf{r} - \frac{Gm_s}{R_v^3}\mathbf{R}_v + \frac{Gm_s}{R^3}\mathbf{R} \\ &= -\frac{Gm_p}{r^3}\mathbf{r}\left(1 + \frac{m_v}{m_p}\right) - \frac{Gm_s}{R_v^3}\left[\mathbf{R}_v - \left(\frac{R_v}{R}\right)^3\mathbf{R}\right].\end{aligned} \tag{6.52}$$

Substituting $\mathbf{R}_v$ from Eq. (6.39) into Eq. (6.52) gives

$$\begin{aligned}{}^{\mathcal{I}}\mathbf{a}_{v/p} &= -\frac{Gm_p}{r^3}\mathbf{r}\left(1 + \frac{m_v}{m_p}\right) - \frac{Gm_s}{R_v^3}\left[\mathbf{R} + \mathbf{r} - \left(\frac{R_v}{R}\right)^3\mathbf{R}\right] \\ &= -\frac{Gm_p}{r^3}\mathbf{r}\left(1 + \frac{m_v}{m_p}\right) - \frac{Gm_s}{R_v^3}\left\{\mathbf{r} + \left[1 - \left(\frac{R_v}{R}\right)^3\right]\mathbf{R}\right\}.\end{aligned} \tag{6.53}$$

Observe now that $m_v \ll m_p$ which implies that $m_v/m_p \approx 0$ from which the inertial acceleration of the spacecraft relative to the planet in Eq. (6.53) is approximated as

$${}^{\mathcal{I}}\mathbf{a}_{v/p} = -\frac{Gm_p}{r^3}\mathbf{r} - \frac{Gm_s}{R_v^3}\left\{\mathbf{r} + \left[1 - \left(\frac{R_v}{R}\right)^3\right]\mathbf{R}\right\}. \tag{6.54}$$

Furthermore, because the distance from the Sun to the spacecraft is approximately the same as the distance from the Sun to the planet, that is, $R_v \approx R$, the term $1-(R_v/R)^3 \approx 0$. The inertial acceleration of the spacecraft relative to the planet in Eq. (6.54) is then approximated further as

$${}^{\mathcal{I}}\mathbf{a}_{v/p} = -\frac{Gm_p}{r^3}\mathbf{r} - \frac{Gm_s}{R^3}\mathbf{r} = \mathbf{a}_p + \mathbf{p}_s, \tag{6.55}$$

where the quantities $\mathbf{a}_p$ and $\mathbf{p}_s$ are defined as

$$\mathbf{a}_p = -\frac{Gm_p}{r^3}\mathbf{r}, \tag{6.56}$$

$$\mathbf{p}_s = -\frac{Gm_s}{R^3}\mathbf{r}. \tag{6.57}$$

It is noted that, unlike Eq. (6.44), where the Sun was considered the primary body and the planet was considered the perturbing body, in Eq. (6.55) the planet is considered the primary body and the Sun is considered the perturbing body. The magnitudes of $\mathbf{a}_p$ and $\mathbf{p}_s$ are then given as

$$a_p = \|\mathbf{a}_p\| = \frac{Gm_p}{r^2}, \tag{6.58}$$

$$p_s = \|\mathbf{p}_s\| = \frac{Gm_s}{R^3}r. \tag{6.59}$$

Now, for motion where the Sun is considered the primary body and the planet is considered the secondary body, the ratio A_s/P_p is considered the deviation from two-body motion relative to the Sun. Similarly, for motion where the planet is considered the primary body and the Sun is considered the secondary body, the ratio a_p/p_s is considered the deviation from two-body motion relative to the planet. Then, the planet will have less influence on the motion of the spacecraft if

$$\frac{p_s}{a_p} < \frac{P_p}{A_s}. \tag{6.60}$$

Substituting the results of Eqs. (6.48), (6.49), (6.58), (6.59) into Eq. (6.60) gives

$$\frac{Gm_s r/R^3}{Gm_p/r^2} < \frac{Gm_p/r^2}{Gm_s/R^2} \tag{6.61}$$

which can be re-written as

$$\frac{m_s}{m_p}\left(\frac{r}{R}\right)^3 < \frac{m_p}{m_s}\left(\frac{R}{r}\right)^2. \tag{6.62}$$

Rearranging Eq. (6.62) gives

$$\left(\frac{r}{R}\right)^5 < \left(\frac{m_p}{m_s}\right)^2. \tag{6.63}$$

The sphere of influence of a planet, denoted r_{SOI}, is then defined as the condition where the two sides of Eq. (6.63) are equal, that is, the sphere of influence is given as

$$\left(\frac{r_{\text{SOI}}}{R}\right)^5 = \left(\frac{m_p}{m_s}\right)^2 \tag{6.64}$$

which gives

$$r_{\text{SOI}} = R\left(\frac{m_p}{m_s}\right)^{2/5}. \tag{6.65}$$

6.5 Patched-Conic Approximation

In Chapters 1–5 the focus was on the motion of a spacecraft under the influence of a single central body. In the case of interplanetary orbital transfer, however, the spacecraft moves under the influence of more than one body. A starting point for designing interplanetary orbit transfers is to employ a technique where only one central body is considered during different segments or phases of the transfer and then to piece together an interplanetary transfer from these pieces. Such an approximation is called a *patched conic approximation* because it utilizes the results of Chapters 1–5 where a spacecraft is under the influence of a single central body moves in the path defined by a conic section (that is, an ellipse, parabola, or hyperbola). The patched-conic approximation consists of three distinct phases: (1) leaving sphere of influence of the departure planet; (2) an elliptic heliocentric transfer orbit that starts at the perihelion of the transfer orbit and terminates at the apohelion of the transfer orbit; and (3) entering the sphere of influence of the arrival planet. The first and third phases have a somewhat opposite nature in that the planetary departure changes the orbit relative to the departure planet from elliptic to hyperbolic while planetary arrival changes the orbit relative to the arrival planet from hyperbolic to elliptic. In the next three sections these various phases of flight are considered along with conditions that define the transitions between the phases. A fundamental assumption made in the patched-conic approximation considered here is that the transfer occurs in the ecliptic plane, that is, the Sun, the departure planet, the arrival planet, and the spacecraft are all moving in the same plane. Finally, in this section only a transfer from an inner planet to an outer planet is considered because the geometry for a transfer from an outer planet to an inner planet is exactly opposite the geometry of a transfer from an inner planet to an outer planet.

6.5.1 Planetary Departure

Suppose that we consider the transfer from a relative inner planet to a relative outer planet. The starting planet is called the *departure planet* while the arrival planet is called the *arrival planet*. A departure from an inner planet to an outer planet is shown schematically in Fig. 6.3(a). Now consider a spacecraft in a circular orbit, denoted a *parking orbit*, relative to the departure planet. In order for the spacecraft to be transferred to a arrival planet it is necessary for the spacecraft to be placed onto a heliocentric transfer orbit. Consequently, it is necessary that the spacecraft escape the gravitational field of the departure planet which requires that the spacecraft be placed onto a hyperbolic orbit relative to the departure planet. Let ${}^{\mathcal{I}}\mathbf{v}_\infty$ be the hyperbolic excess inertial velocity and let $v_\infty = \|{}^{\mathcal{I}}\mathbf{v}_\infty\|$ be the corresponding hyperbolic excess speed. In other words, v_∞ is the speed in excess of parabolic speed as the vehicle approaches the asymptote defined by the hyperbola (see Fig. 1.8 on page 27 of Chapter 1). Next, let ${}^{\mathcal{I}}\mathbf{V}_{c1}$ be the inertial velocity of the departure planet (relative to the Sun). It is seen from Fig. 6.3(a) that in order for the spacecraft to be placed onto a heliocentric Hohmann transfer from the departure planet to the arrival planet, it is necessary that ${}^{\mathcal{I}}\mathbf{v}_\infty$ be parallel to and in the same direction as ${}^{\mathcal{I}}\mathbf{V}_{c1}$. It is also seen from Fig. 6.3(a) that the spacecraft must leave the departure planet from the anterior of the sphere of influence. Next, the hyperbolic excess v_∞ must be chosen such that the perihelion and apohelion are R_1 and R_2, respectively. In other words, the value of v_∞ must correspond to the speed required to place the spacecraft onto an elliptic heliocentric transfer transfer orbit whose perihelion is the radius of the circular heliocentric orbit of the departure planet and whose apohelion is the radius of the circular heliocentric orbit of the arrival planet. Therefore, the departure excess hyperbolic speed along the departure hyperbolic orbit, v_∞, is given as

$$v_\infty = \sqrt{\frac{2\mu_s}{R_1} - \frac{2\mu_s}{R_1 + R_2}} - \sqrt{\frac{\mu_s}{R_1}} = \sqrt{\frac{\mu_s}{R_1}}\left(\sqrt{\frac{2R_2}{R_1 + R_2}} - 1\right), \tag{6.66}$$

where μ_s is the gravitational parameter of the Sun. The periapsis radius from the center of the departure planet is then obtained from Eqs. (1.139) and (1.140) on page 26 and Eq. (1.43) on page 13 of Chapter 1 as

$$r_p = a(1-e) = \frac{p}{1-e^2}(1-e) = \frac{p}{1+e} = \frac{h^2/\mu_1}{1+e}, \tag{6.67}$$

where μ_1 is the gravitational parameter of the orginating planet and the quantities h, p and e are measured relative to the planet. Equation (6.67) can be solved for h^2 to give

$$h^2 = a(1-e)(1+e)\mu_1 = a(1-e^2)\mu_1. \tag{6.68}$$

Next, from Eq. (1.66), the specific mechanical energy on the departure orbit is given as

$$\mathcal{E} = \frac{{}^{\mathcal{I}}\mathbf{v} \cdot {}^{\mathcal{I}}\mathbf{v}}{2} - \frac{\mu}{r} = -\frac{\mu_1}{2a}. \tag{6.69}$$

Noting as the spacecraft approaches the asymptote on the outbound hyperbolic the distance from the departure planet approaches ∞ which implies that $r_\infty = \infty$. Evaluating Eq. (6.69) at $r = r_\infty$ then gives

$$\mathcal{E} = \frac{v_\infty^2}{2} - \frac{\mu}{r_\infty} = \frac{v_\infty^2}{2} = -\frac{\mu_1}{2a}. \tag{6.70}$$

Solving Eq. (6.70) for v_∞^2 gives

$$v_\infty^2 = -\frac{\mu_1}{a}. \tag{6.71}$$

It is noted in Eq. (6.71) that $a < 0$ on a hyperbolic which implies that $v_\infty^2 > 0$. Rearranging Eq. (6.71), the semi-major axis of the hyperbolic orbit relative to the departure planet is given as

$$a = -\frac{\mu_1}{v_\infty^2}. \tag{6.72}$$

Substituting the value for a in Eq. (6.72) into Eq. (6.68) gives

$$h^2 = -\frac{\mu_1}{v_\infty^2}(1-e^2)\mu_1 = -\frac{\mu_1^2(1-e^2)}{v_\infty^2} = \frac{\mu_1^2(e^2-1)}{v_\infty^2} \tag{6.73}$$

It is note that, because the orbit relative to the departure planet is hyperbolic, the value h^2 in Eq. (6.73) must be greater than zero. Substituting the expression for h^2 given in Eq. (6.73) into Eq. (6.67) gives

$$r_p = \frac{\frac{\mu_1^2(e^2-1)/\mu_1}{v_\infty^2}}{1+e} = \frac{\mu_1(e-1)}{v_\infty^2}. \tag{6.74}$$

Solving Eq. (6.74) for the eccentricity gives

$$e = 1 + \frac{r_p v_\infty^2}{\mu_1}. \tag{6.75}$$

Furthermore, substituting the value of e given in Eq. (6.75) into Eq. (6.73) gives

$$\begin{aligned} h^2 &= \frac{\mu_1^2}{v_\infty^2}\left[\left(1 + \frac{r_p v_\infty^2}{\mu_1}\right)^2 - 1\right] = \frac{\mu_1^2}{v_\infty^2}\left[1 + \frac{2r_p v_\infty^2}{\mu_1} + \left(\frac{r_p v_\infty^2}{\mu_1}\right)^2 - 1\right] \\ &= \frac{\mu_1^2}{v_\infty^2}\left[\frac{2r_p v_\infty^2}{\mu_1} + \left(\frac{r_p v_\infty^2}{\mu_1}\right)^2\right] = \frac{\mu_1^2}{v_\infty^2}\frac{r_p v_\infty^2}{\mu_1}\left(2 + \frac{r_p v_\infty^2}{\mu_1}\right) \\ &= \mu_1 r_p\left(2 + \frac{r_p v_\infty^2}{\mu_1}\right) = \mu_1 r_p \frac{r_p}{\mu_1}\left(v_\infty^2 + \frac{2\mu_1}{r_p}\right) = r_p^2\left(v_\infty^2 + \frac{2\mu_1}{r_p}\right) \end{aligned} \tag{6.76}$$

The magnitude of the specific angular momentum along the hyperbolic orbit relative to the departure planet is then obtained from Eq. (6.76) as

$$h = r_p\sqrt{v_\infty^2 + \frac{2\mu_1}{r_p}}. \tag{6.77}$$

Now it is noted that at periapsis relative to the departure planet the position of the spacecraft and the inertial velocity of the spacecraft are othogonal to one another (that is, $\mathbf{r}\cdot{}^{\mathcal{I}}\mathbf{v}$ is zero at periapsis of the departure planet), the magnitude of the specific angular momentum at periapsis of the departure planet is given as

$$h = r_p v_p. \tag{6.78}$$

Equation (6.78) implies that

$$v_p = \frac{h}{r_p} = \sqrt{v_\infty^2 + \frac{2\mu_1}{r_p}}. \tag{6.79}$$

The magnitude of the escape impulse is then obtained as follows. Recall that the spacecraft starts in a circular parking orbit of radius r_p relative to the departure planet. Therefore, the speed of the spacecraft relative to the departure planet the instant before the escape impulse is applied is given as

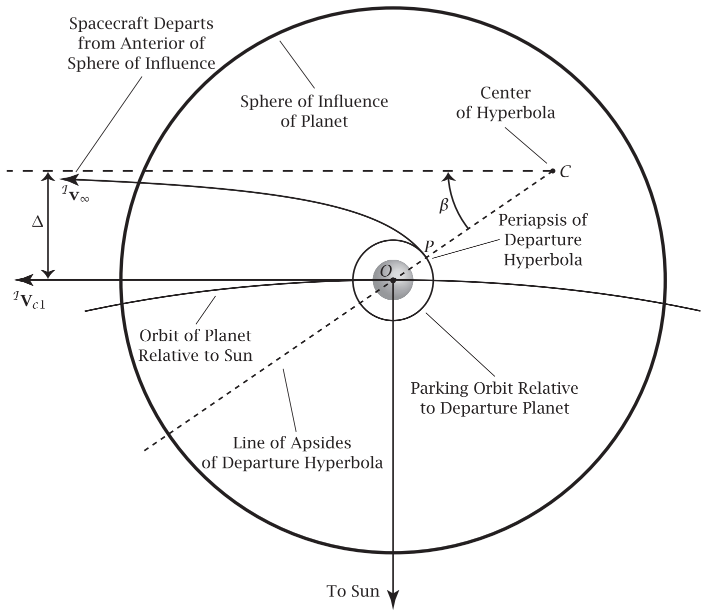
$$v_{c1} = \sqrt{\frac{\mu_1}{r_p}}. \tag{6.80}$$

The escape impulse is then given as the difference between v_p of Eq. (6.79) and Eq. (6.80), that is,

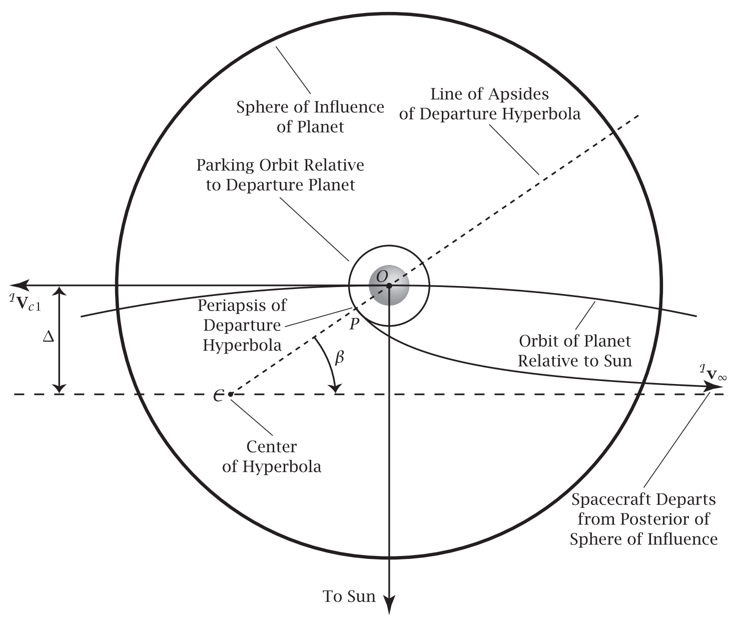
$$\Delta v_{\text{escape}} = v_p - v_{c1} = \sqrt{v_\infty^2 + \frac{2\mu_1}{r_p}} - \sqrt{\frac{\mu_1}{r_p}} = v_{c1}\left[\sqrt{2 + \left(\frac{v_\infty}{v_{c1}}\right)^2} - 1\right]. \tag{6.81}$$

Finally, the angle β that defines the direction of the departure asymptote relative to the line of apsides of the departure hyperbola and also defined the location on the departure hyperbola (where Δv_{escape} must be applied) is given from Eq. (1.135) on page 26 of Chapter 1 as

$$\beta = \cos^{-1}\left(\frac{1}{e}\right). \tag{6.82}$$



(a) Departure from inner planet to outer planet.



(b) Departure from outer planet to inner planet.

Figure 6.3 Departure of a spacecraft from an inner planet to a outer planet and a relative outer planet to a relative inner planet.

6.5.2 Planetary Arrival

Continuing with an interplanetary transfer from a relative inner planet to a relative outer planet, suppose now that the arrival of the spacecraft at the relative outer planet is considered. The terminating planet is called the *arrival planet*. An arrival at a relative outer planet is shown schematically in Fig. 6.4(a). In order to place the spacecraft in a terminal orbit relative to the arrival planet it is necessary that the spacecraft be captured by the gravitational field of the arrival planet. This capture requires arrive at the sphere of influence of the arrival planet on a hyperbolic trajectory relative to the arrival planet and that an impulse be applied so that the spacecraft can be captured by the arrival planet. Similar to planetary departure, let ${}^{\mathcal{I}}\mathbf{v}_\infty$ be the hyperbolic excess inertial velocity and let $v_\infty = \|{}^{\mathcal{I}}\mathbf{v}_\infty\|$ be the corresponding hyperbolic excess speed. In other words, v_∞ is the speed in excess of parabolic speed as the vehicle approaches the arrival planet along the asymptote defined by the hyperbola (see Fig. 1.8 on page 27 of Chapter 1). Next, let ${}^{\mathcal{I}}\mathbf{V}_{c2}$ be the inertial velocity of the arrival planet (relative to the Sun). It is seen from Fig. 6.4(a) that ${}^{\mathcal{I}}\mathbf{v}_\infty$ must be parallel to and in the opposite direction as ${}^{\mathcal{I}}\mathbf{V}_{c2}$. In other words, in order for the spacecraft to arrive at the outer (arrival) planet it is necessary that the planet be overtaking the spacecraft, thus making it such that the spacecraft is moving toward the planet upon arrival. It is also seen from Fig. 6.4(a) that the spacecraft must arrive at the arrival planet from the anterior of the sphere of influence. Therefore, the arrival excess hyperbolic speed along the arrival hyperbolic orbit, v_∞, is given as

$$v_\infty = \sqrt{\frac{\mu_s}{R_2}} - \sqrt{\frac{2\mu_s}{R_2} - \frac{2\mu_s}{R_1 + R_2}} = \sqrt{\frac{\mu_s}{R_2}}\left(1 - \sqrt{\frac{2R_1}{R_1 + R_2}}\right), \tag{6.83}$$

where μ_s is the gravitational parameter of the Sun. The periapsis radius from the center of the arrival planet is then obtained from Eqs. (1.139) and (1.140) on page 26 and Eq. (1.43) on page 13 of Chapter 1 as

$$r_p = a(1 - e) = \frac{p}{1 - e^2}(1 - e) = \frac{p}{1 + e} = \frac{h^2/\mu_2}{1 + e}, \tag{6.84}$$

where μ_2 is the gravitational parameter of the arrival planet and the quantities h, p and e are measured relative to the arrival planet. Solving Eq. (6.84) for the magnitude of the specific angular momentum relative to the planet gives

$$h = \frac{\mu_2\sqrt{e^2 - 1}}{v_\infty}, \tag{6.85}$$

where it is noted that $e > 1$ because the spacecraft is on an incoming hyperbolic trajectory relative to the arrival planet. Substituting the expression for h as given in Eq. (6.85) into the expression for the periapsis radius in Eq. (6.84), the eccentricity of the arrival hyperbola is obtained as

$$e = 1 + \frac{r_p v_\infty^2}{\mu_2}. \tag{6.86}$$

Furtheremore, substituting the value of e given in Eq. (6.86) into Eq. (6.85), the magnitude of the specific angular momentum along the arrival hyperbolic orbit relative to the departure planet is given a

$$h = r_p\sqrt{v_\infty^2 + \frac{2\mu_2}{r_p}}. \tag{6.87}$$

Now it is noted that at periapsis relative to the arrival planet the position of the spacecraft and the inertial velocity of the spacecraft are othogonal to one another (that is, $\mathbf{r} \cdot {}^{\mathcal{I}}\mathbf{v}$ is zero at periapsis of the arrival planet), the magnitude of the specific angular momentum at periapsis of the arrival planet is given as

$$h = r_p v_p. \tag{6.88}$$

Equation (6.88) implies that

$$v_p = \frac{h}{r_p} = \sqrt{v_\infty^2 + \frac{2\mu_2}{r_p}}. \tag{6.89}$$

The magnitude of the capture impulse is then obtained as follows. First, the speed of the spacecraft in the terminal circular orbit relative to the arrival planet is given as

$$v_{c2} = \sqrt{\frac{\mu_2}{r_p}}. \tag{6.90}$$

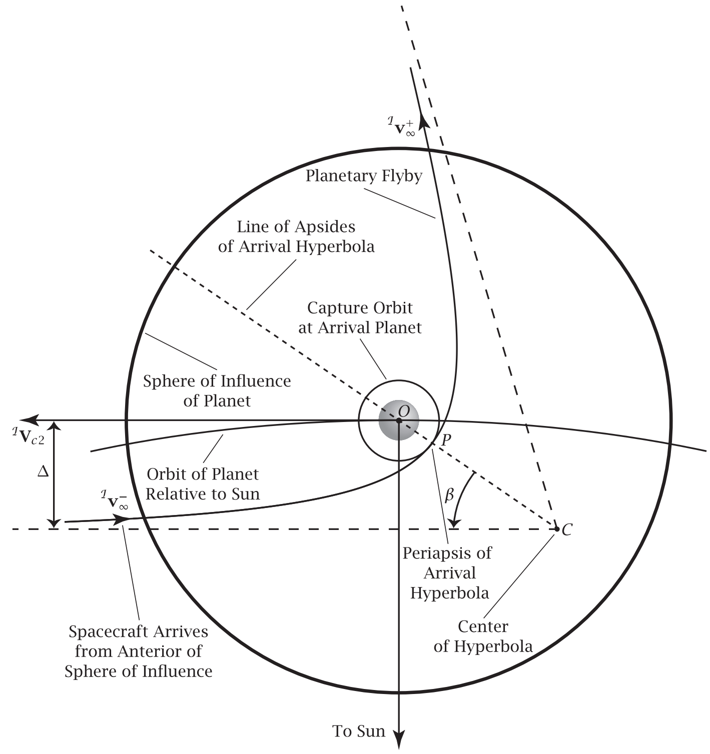
The capture impulse is then given as the difference between v_p of Eq. (6.79) and Eq. (6.80), that is,

$$\Delta v_{\text{capture}} = v_p - v_{c2} = \sqrt{v_\infty^2 + \frac{2\mu_2}{r_p}} - \sqrt{\frac{\mu_2}{r_p}} = v_{c2}\left[\sqrt{2 + \left(\frac{v_\infty}{v_{c2}}\right)^2} - 1\right] \tag{6.91}$$

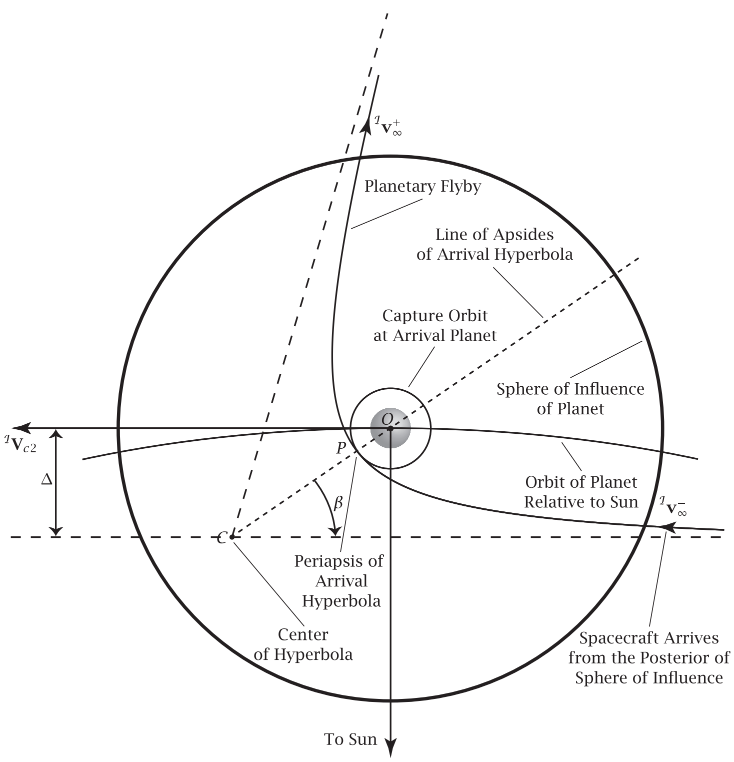
Now, the angle β that defines the direction of the arrival asymptote relative to the line of apsides of the arrival hyperbola and also defines the location of the periapsis on the arrival hyperbola (where $\Delta v_{\text{capture}}$ must be applied) is given from Eq. (1.135) on page 26 of Chapter 1 as

$$\beta = \cos^{-1}\left(\frac{1}{e}\right). \tag{6.92}$$

Finally, suppose that the capture impulse $\Delta v_{\text{capture}}$ is *not* applied as the vehicle moves through the sphere of influence of the arrival planet. Then, as shown in Fig. 6.4(a), instead of being captured by the arrival planet, the spacecraft will exit the sphere of influence with a speed v_∞ but in the direction defined by the outbound asymptote. In other words, if no capture impulse is applied, the spacecraft will undergo a planetary flyby of what would have been the arrival planet. If the geometry is as shown in Fig. 6.4(a), the spacecraft will exit the sphere of influence of the arrival planet in such a manner that it will be sent further from the Sun than it was upon entering the sphere of influence of the arrival planet.



(a) Arrival at an outer planet from an inner planet.



(b) Arrival at an inner planet from an outer planet.

Figure 6.4 Arrival of a spacecraft at an inner planet from a outer planet and and at an outer planet from an inner planet.

6.5.3 Planetary Flyby (Gravity Assist) Trajectories

Suppose now that the case of a planetary flyby is considered. Similar to planetary arrival, a planetary flyby is when a spacecraft enters the sphere of influence of a planet. Unlike the case of planetary arrival, however, when the spacecraft performs a planetary flyby the orbit of the spacecraft remains hypebolic and eventually leaves the sphere of influence of the planet. A planetary flyby is shown in Figs. 6.4(a) and 6.4(b) for the cases where a spacecraft approaches an outer planet from an inner planet (Fig. 6.4(a)) and approaches an inner planet from an outer planet (Fig. 6.4(b)). In either case, the velocity of the spacecraft relative to the planet on the inboud and outbound hyperbolic asymptotes are ${}^{\mathcal{I}}\mathbf{v}_\infty^-$ and ${}^{\mathcal{I}}\mathbf{v}_\infty^+$, respectively, as shown in Figs. 6.4(a) and 6.4(b). Next, the turn angle, denoted δ, of a planetary flyby is shown in Fig. 6.5, where the turn angle is the change in direction from ${}^{\mathcal{I}}\mathbf{v}_\infty^-$ to ${}^{\mathcal{I}}\mathbf{v}_\infty^+$ (where it is noted again that $\|{}^{\mathcal{I}}\mathbf{v}_\infty^-\| = \|{}^{\mathcal{I}}\mathbf{v}_\infty^+\| = v_\infty$). Denoting ${}^{\mathcal{I}}\mathbf{V}_p$ as the velocity of the planet relative to the Sun, the velocity of the spacecraft relative to the Sun on the inbound hyperbolic asymptote is given as

$$ {}^{\mathcal{I}}\mathbf{v}_s = {}^{\mathcal{I}}\mathbf{V}_p + {}^{\mathcal{I}}\mathbf{v}_\infty^-. \tag{6.93} $$

Similarly, the velocity of the spacecraft relative to the Sun on the outbound hyperbolic asymptote is given as

$$ {}^{\mathcal{I}}\mathbf{v}_s = {}^{\mathcal{I}}\mathbf{V}_p + {}^{\mathcal{I}}\mathbf{v}_\infty^+. \tag{6.94} $$

Assume now that the time taken for the spacecraft to pass through the sphere of influence of the planet is small in comparison to the total transfer time. Then as an approximation it can be taken that the change in velocity ${}^{\mathcal{I}}\mathbf{v}_\infty^+ - {}^{\mathcal{I}}\mathbf{v}_\infty^-$ occurs instantaneously (such that the position of the planet does not change) and the impulse due to the planetary flyby is given as

$$ \Delta^{\mathcal{I}}\mathbf{v}_{fb} = ({}^{\mathcal{I}}\mathbf{V}_p + {}^{\mathcal{I}}\mathbf{v}_\infty^+) - ({}^{\mathcal{I}}\mathbf{V}_p + {}^{\mathcal{I}}\mathbf{v}_\infty^-) = {}^{\mathcal{I}}\mathbf{v}_\infty^+ - {}^{\mathcal{I}}\mathbf{v}_\infty^-. \tag{6.95} $$

Then, given that $\|{}^{\mathcal{I}}\mathbf{v}_\infty^-\| = \|{}^{\mathcal{I}}\mathbf{v}_\infty^+\| = v_\infty$, the magnitude of the impulse given in Eq. (6.95) is given from the law of cosines as

$$ \left(\Delta v_{fb}\right)^2 = \|\Delta^{\mathcal{I}}\mathbf{v}\|^2 = 2v_\infty^2 - 2v_\infty^2\cos\delta = 2v_\infty^2(1-\cos\delta). \tag{6.96} $$

Then, using the fact that $\cos(2\alpha) = 1 - 2\sin^2\alpha$,

$$ 1 - \cos\delta = 2\sin^2\left(\frac{\delta}{2}\right) $$

from which Eq. (6.96) simplifies to

$$ \Delta v_{fb} = 2v_\infty \sin\left(\frac{\delta}{2}\right). \tag{6.97} $$

Recall now that the periapsis radius of the hyperbola is denoted r_p. Suppose further that the radius and gravitational parameter of the planet are denoted, respectively, as R and μ. Then Eq. (6.97) can be written in terms r_p, R, and μ as follows. First, the eccentricity in Eq. (6.86) can be written as

$$ e = 1 + \frac{r_p v_\infty^2}{\mu} = 1 + \frac{r_p}{R}\frac{R}{\mu}v_\infty^2 = 1 + \frac{r_p}{R}\frac{v_\infty^2}{v_s^2}, \tag{6.98} $$

where

$$v_s = \sqrt{\frac{\mu}{R}} \tag{6.99}$$

is the speed of a spacecraft in a circular orbit of radius R. Now, from the geometry of a hyperbola as given in Fig. 1.8 on page 27 of Chapter 1, the turn angle δ is given in terms of the asymptote angle β as

$$\delta + 2\beta = \pi \tag{6.100}$$

which implies that

$$\delta = \pi - 2\beta. \tag{6.101}$$

The turn half-angle, $\delta/2$, is then given as

$$\frac{\delta}{2} = \frac{\pi}{2} - \beta. \tag{6.102}$$

Taking the sine on both sides of Eq. (6.102) gives

$$\sin\left(\frac{\delta}{2}\right) = \sin\left(\frac{\pi}{2} - \beta\right) = \sin\left(\frac{\pi}{2}\right)\cos\beta + \cos\left(\frac{\pi}{2}\right)\sin\beta = \cos\beta. \tag{6.103}$$

Then, using the result of Eq. (6.92),

$$\sin\left(\frac{\delta}{2}\right) = \frac{1}{e}. \tag{6.104}$$

Substituting the result of Eq. (6.98) into Eq. (6.104) gives

$$\sin\left(\frac{\delta}{2}\right) = \frac{1}{1 + \dfrac{r_p}{R}\dfrac{v_\infty^2}{v_s^2}} \tag{6.105}$$

The planetary flyby impulse of Eq. (6.97) is then given as

$$\Delta v_{fb} = \frac{2v_\infty}{1 + \dfrac{r_p}{R}\dfrac{v_\infty^2}{v_s^2}} \tag{6.106}$$

Normalizing the flyby impulse by the circular speed v_s given in Eq. (6.99) gives

$$\frac{\Delta v_{fb}}{v_s} = \frac{2v_\infty/v_s}{1 + \dfrac{r_p}{R}\dfrac{v_\infty^2}{v_s^2}} \tag{6.107}$$

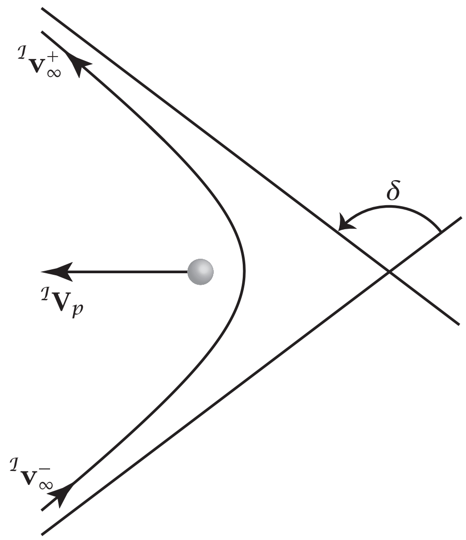


Figure 6.5 Schematic of a planetary flyby (gravity assist) trajectory.

6.5.4 Planetary Flyby Following an Interplanetary Hohmann Transfer

Consider a spacecraft that has been transferred from a departure planet to an arrival planet via an interplanetary Hohmann transfer. Furthermore, suppose that, subsequent to the interplanetary Hohmann transfer, the spacecraft will perform a flyby (gravity assist) of the arrival planet, where the fly will be used to send the spacecraft either further from or closer to the Sun. In addition, assume that the planetary flyby will be performed on the dark side of the planet (that is, the flyby will occur on the side of the planet *opposite* that of the Sun). Because an interplanetary Hohmann transfer has been used to leave the departure planet, the heliocentric inertial velocity of the spacecraft upon entrace to the sphere of influence of the arrival planet, $^{\mathcal{I}}\mathbf{v}^-$, is parallel to the heliocentric inertial velocity, $^{\mathcal{I}}\mathbf{V}_p$, of the planet. The planetary flyby can occur at an outer or an inner arrival planet. Because the conditions for the planetary flyby of an outer or an inner planet are different, each of these two cases are now considered separately.

Figure 6.6(a) provides a schematic of a planetary flyby that follows an interplanetary Hohmann transfer from an inner planet to an outer planet. Moreover, Fig. 6.6(b) provides a schematic of the velocity of the spacecraft relative to the Sun before and after the flyby along with the velocity of the planet. Because $^{\mathcal{I}}\mathbf{v}_v^-$ and $^{\mathcal{I}}\mathbf{V}_p$ are parallel, the pre-flyby heliocentric flight path angle is zero while the post-flyby heliocentric flight path angle is negative. Also, because the transfer is to an outer planet, the heliocentric speed of the spacecraft is smaller than the heliocentric speed of the planet prior to the flyby, that is,

$$v_{v/s}^- = \|^{\mathcal{I}}\mathbf{v}^-\| < V_p. \tag{6.108}$$

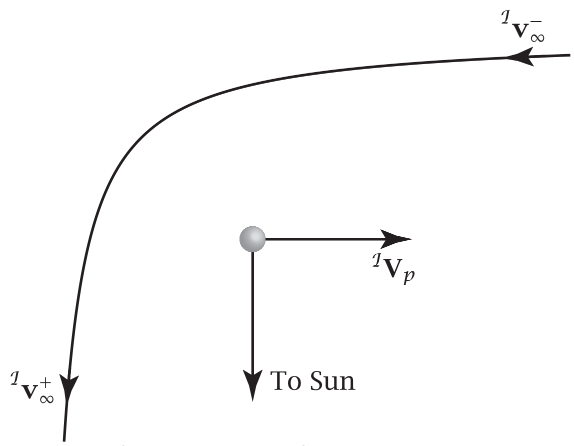
Consequently, as shown in Fig. 6.6(b), the pre-flyby hyperbolic excess velocity relative to the planet, $\|^{\mathcal{I}}\mathbf{v}_\infty^-\|$, lies in the direction opposite the heliocentric velocity of the planet. The hyperbolic excess speed upon entrance to the sphere of influence of the outer planet is then given as

$$v_\infty^- = V_p - v^-. \tag{6.109}$$

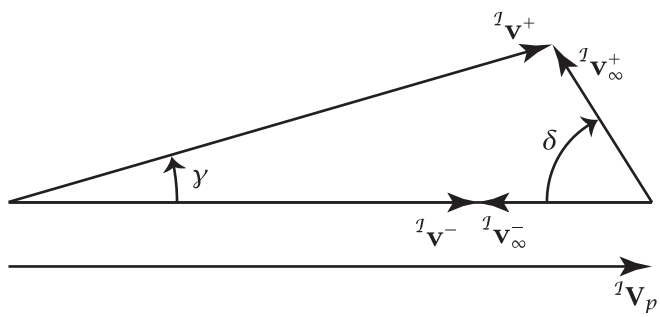
Using Fig. 6.6(b) together with the fact that $v_\infty = v_\infty^+ = v_\infty^- = \|{}^{\mathcal{I}}\mathbf{v}_\infty^-\|$ and denoting the post-flyby heliocentric inertial speed of the spacecraft by v^+, the turn angle of the flyby can then be computed from the law of cosines as

$$(v^+)^2 = v_\infty^2 + V_p^2 - 2V_p v_\infty \cos\delta, \tag{6.110}$$

where v^+ is the heliocentric speed of the spacecraft following the flyby.



(a) Flyby of an outer planet following an interplanetary Hohmann transfer.



(b) Turn angle of an outer planet following an interplanetary Hohmann transfer.

Figure 6.6 Schematic of an outer planet flyby and associated turn angle of an inner planet flyby following an interplanetary Hohmann transfer.

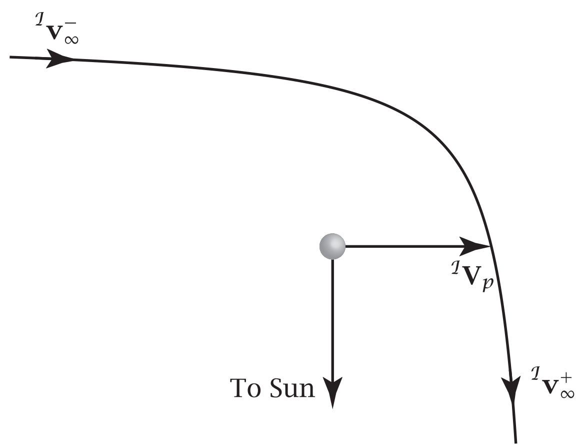
Next, Fig. 6.7(a) provides a schematic of a planetary flyby that follows an interplanetary Hohmann transfer from an outer planet to an inner planet. Moreover, Fig. 6.7(b) provides a schematic of the velocity of the spacecraft relative to the Sun before and after the flyby along with the velocity of the planet. Because ${}^{\mathcal{I}}\mathbf{v}^-$ and ${}^{\mathcal{I}}\mathbf{V}_p$ are parallel, the pre-flyby heliocentric flight path angle is zero while the post-flyby heliocentric flight path angle is negative. Also, because the transfer is to an inner planet, the pre-flyby heliocentric inertial speed of the spacecraft, $v^- = \|{}^{\mathcal{I}}\mathbf{v}^-\|$, is larger than the heliocentric speed of the planet prior to the flyby, that is, $v^- > V_p$. Consequently, as shown in Fig. 6.7(b), the pre-flyby hyperbolic excess velocity relative to the planet, $\|{}^{\mathcal{I}}\mathbf{v}_\infty^-\|$, lies in the same direction as the heliocentric velocity of the planet. The hyperbolic excess speed upon entrance to the sphere of influence of the outer planet is then given as

$$v_\infty^- = v^- - V_p. \tag{6.111}$$

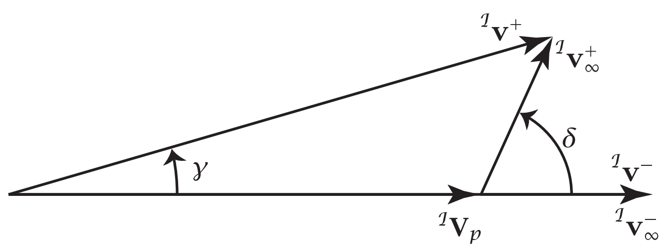
Using Fig. 6.7(b) together with the fact that $v_\infty = v_\infty^+ = v_\infty^- = \|{}^{\mathcal{I}}\mathbf{v}_\infty^-\|$ and denoting the post-flyby heliocentric inertial speed of the spacecraft by v^+, the turn angle of the flyby can then be computed from the law of cosines as

$$(v^+)^2 = v_\infty^2 + V_p^2 - 2V_p v_\infty \cos(\pi - \delta) = v_\infty^2 + V_p^2 + 2V_p v_\infty \cos\delta, \tag{6.112}$$

where the identity $\cos(\pi - \delta) = \cos\pi\cos\delta + \sin\pi\sin\delta = -\cos\delta$ has been used in Eq. (6.112).



(a) Flyby from an outer planet to inner planet.



(b) Turn angle of a flyby from an outer planet to inner planet.

Figure 6.7 Schematic of an inner planet flyby and associated turn angle of an inner planet flyby following an interplanetary Hohmann transfer.

The results developed in this section can now be used to compute, the turn angle, δ, the periapsis radius, r_p, and the eccentricity, e, of the hyperbolic flyby orbit under the assumption that the post-flyby heliocentric inertial speed, v^+, is specified. First, as given earlier, let R_1 and R_2 be the distances from the Sun to the departure and arrival planets, respectively. Then the heliocentric inertial speed of the spacecraft upon planetary arrival is given as

$$v^- = \sqrt{\frac{2\mu_s}{R_2} - \frac{2\mu_s}{R_1 + R_2}}, \tag{6.113}$$

where it is noted that $(R_1 + R_2)/2$ is the semi-major axis of the heliocentric elliptic interplanetary Hohmann transfer orbit. Next, the heliocentric inertial speed of the planet is given as

$$V_p = \sqrt{\frac{\mu_s}{R_2}} \tag{6.114}$$

which implies that the hyperbolic excess speed is

$$v_\infty = \left| V_p - v^- \right|, \tag{6.115}$$

where the absolute value is used v_∞ must be nonnegative (that is, the speed is a nonnegative quantity). The turn angle is then computed by solving either Eq. (6.110) (in the case of an outer planet flyby) or (6.112) (in the case of an inner planet flyby) for δ. Once the turn angle is computed, the eccentricity of the orbit is obtained by solving Eq. (6.104) for e. Finally, using the value for e obtained from Eq. (6.104), the periapsis radius is obtained by solving Eq. (6.86) for r_p. Note that the post-flyby heliocentric inertial speed v^+ must be known because the objective of the flyby is to alter the heliocentric orbit of the spacecraft in such a manner that the spacecraft is either moving closer or further from the Sun. Thus, as stated, in order to employ the aforementioned procedure it is necessary that the post-flyby heliocentric inertial speed, v^+, be known.

Problems for Chapter 6

6–1 A spacecraft starts in a 350 km altitude orbit relative to the Earth. The objective is to transfer the spacecraft to a circular orbit relative to Mars with an altitude of 350 km. Determine

(a) the impulse required to place the spacecraft onto the departure hyperbola that enables the spacecraft to arrive at an apohelion that is the same distance from the Sun as Mars.

(b) the angle β between the line of apsides of the departure hyperbola and the hyperbolic excess velocity, ${}^{\mathcal{I}}\mathbf{v}_{\infty}$.

(c) the impulse required to insert the spacecraft into the final orbit relative to Mars.

Assume in your answers that the gravitational parameters of Earth and the Sun are given, respectively, as $\mu_e = 3.986 \times 10^5\ \text{km}^3 \cdot \text{s}^{-2}$ and $\mu_s = 1.327 \times 10^{11}\ \text{km}^3 \cdot \text{s}^{-2}$ while the distance from the Sun to Earth and the Sun to Mars are given, respectively, as $R_{es} = 1.496 \times 10^8$ km and $R_{ms} = 2.279 \times 10^8$ km

6–2 A spacecraft starts in a 300 km altitude orbit relative to the Earth. The objective is to transfer the spacecraft to a circular orbit relative to Venus with an altitude of 300 km. Determine

(a) the impulse required to place the spacecraft onto the departure hyperbola that enables the spacecraft to arrive at a perihelion that is the same distance from the Sun as Venus.

(b) the angle β between the line of apsides of the departure hyperbola and the hyperbolic excess velocity, ${}^{\mathcal{I}}\mathbf{v}_{\infty}$.

(c) the impulse required to insert the spacecraft into the final orbit relative to Venus.

Assume in your answers that the gravitational parameters of Earth and the Sun are given, respectively, as $\mu_e = 3.986 \times 10^5\ \text{km}^3 \cdot \text{s}^{-2}$ and $\mu_s = 1.327 \times 10^{11}\ \text{km}^3 \cdot \text{s}^{-2}$ while the distance from the Sun to Earth and the Sun to Venus are given, respectively, as $R_{es} = 1.496 \times 10^8$ km and $R_{ms} = 1.082 \times 10^8$ km

6–3 A spacecraft sends a rover to Mars from Earth. The rover will spend time collecting samples on Mars before a spacecraft is launched from Mars to return the samples collected by the rover back to Earth. Suppose that rover arrived at Mars on 1 January 2018 at 00:00 Greenwich Mean Time (henceforth referred to as the *Epoch*). Determine the following information

(a) the mean motion of both Earth and Mars;

(b) the phase angle between Earth and Mars at the Epoch;

(c) the phase angle required at departure from Mars;

(d) the time available to the rover to collect samples before it is possible to perform a Hohmann transfer to return the samples collected by the rover back to Earth;

(e) the impulses required to accomplish the orbit transfer;

In solving this question, determine the precise locations of Earth and Mars on the day the astronauts leave Mars given that they arrived at Mars at the Epoch given.

6–4 A spacecraft arrives at Venus from Earth using a Hohmann transfer from Earth. The objective is to perform a planetary flyby of Venus so that the spacecraft can be sent to Mercury (that is, the flyby will taken the spacecraft closer to the Sun). The flyby will be designed so that the elliptic heliocentric orbit after the flyby has a heliocentric orbital period that is $5/2$ the orbital period of Mercury. Determine the following information:

(a) the semi-major axis of the heliocentric orbit of the spacecraft after the flyby;

(b) the turn-angle requires to perform the planetary flyby;

(c) the altitude at the periapsis of Venus that corresponds to the turn-angle computed in part (b);

(d) the eccentricity of the heliocentric orbit of the spacecraft after the flyby;

(e) whether or not it is possible for the spacecraft to cross the orbit of Mercury.