Pseudospectral Methods for Solving Infinite-Horizon Optimal Control Problems

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Abstract

An important aspect of numerically approximating the solution of an infinite-horizon optimal control problem is the manner in which the horizon is treated. Generally, an infinite-horizon optimal control problem is approximated with a finite-horizon problem. In such cases, regardless of the finite duration of the approximation, the final time lies an infinite duration from the actual horizon at $t = +\infty$. In this paper we describe two new direct pseudospectral methods using Legendre-Gauss (LG) and Legendre-Gauss-Radau (LGR) collocation for solving infinite-horizon optimal control problems numerically. A smooth, strictly monotonic transformation is used to map the infinite time domain $t \in [0, \infty)$ onto a half-open interval $\tau \in [-1, 1)$. The resulting problem on the finite interval is transcribed to a nonlinear programming problem using collocation. The proposed methods yield approximations to the state and costate on the entire horizon, including approximations at $t = +\infty$. These pseudospectral methods can be written equivalently in either a differential or an implicit integral form. In numerical experiments, the discrete solution exhibits exponential convergence as a function of the number of collocation points. It is shown that the map $\phi : [-1, +1) \to [0, +\infty)$ can be tuned to improve the quality of the discrete approximation.

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1 Introduction

Over the last decade, pseudospectral methods have become increasingly popular in the numerical solution of optimal control problems.[1–10] Pseudospectral methods are a class of *direct collocation* methods where the optimal control problem is transcribed to a nonlinear programming problem (NLP) by parameterizing the state and control using global polynomials and collocating the differential-algebraic equations using nodes obtained from a Gaussian quadrature. The three most commonly used sets of collocation points are *Legendre-Gauss* (LG), *Legendre-Gauss-Radau* (LGR), and *Legendre-Gauss-Lobatto* (LGL) points. These three sets of points are obtained from the roots of a Legendre polynomial and/or linear combinations of a Legendre polynomial and its derivatives. All three sets of points are defined on the domain $[-1, 1]$, but differ significantly in that the LG points include *neither* of the endpoints, the LGR points include *one* of the endpoints, and the LGL points include *both* of the endpoints. In recent years, the two most well-documented pseudospectral methods are the *Legendre-Gauss-Lobatto pseudospectral method*[1] and the *Legendre-Gauss pseudospectral method*.[6,9,10]

In this paper we describe two new pseudospectral methods for the numerical solution of nonlinear infinite-horizon optimal control problems based on either LG or LGR collocation. For either scheme, a smooth, strictly monotonic change of variables is used to map the domain of the infinite time interval $t \in [0, \infty)$ to a finite half open time interval $\tau \in [-1, +1)$. The resulting finite horizon problem is discretized using either LG or LGR collocation. Our collocation schemes avoid the singularity at $\tau = +1$ introduced by the change of variables. Furthermore, in the LG scheme, an explicit formula is derived to compute the state at the horizon (that is, at $t = +\infty$), while in the LGR scheme the state at $t = +\infty$ is a variable in the NLP. Thus, either scheme developed in this paper yields an estimate for the state on the *entire* horizon. In addition, we also present the transformed adjoint systems that relate the Lagrange multipliers of the NLP to the costates of the continuous control problem.

It is noted that an LGR pseudospectral method for approximating the solution of nonlinear infinite-horizon optimal control problems has been previously developed in Ref. 4. The two methods presented in this paper are, however, *fundamentally different* from the method of Ref. 4. In the approach of Ref. 4, the state at the horizon is not a variable in the discrete scheme, and hence, would need to be estimated in another step. Furthermore, the differentiation matrix for the method in Ref. 4 is square and singular whereas the differentiation matrices for the methods derived in this paper are *rectangular* and *full rank*. Consequently, the methods derived in this paper can be written equivalently in *either* a differential or an integrated form.[8,9]

This paper is organized as follows. In Section 2 we describe the LG and LGR collocation points. In Section 3 we state the infinite-horizon optimal control problem. In Section 4 we provide a description of our notation. Section 5 describes our Gauss and Radau pseudospectral methods for solving infinite-horizon optimal control problems. In addition, we show that the first-order optimality conditions associated with our Gauss and Radau methods are equivalent to a pseudospectral schemes for the continuous costate equation. In Section 6 we demonstrate the method on an example. In Section 7 we provide further details about how our infinite-horizon methods differ from the method of Ref. 4. Finally, in Section 8 we provide conclusions.

2 LG and LGR Collocation Points

In this paper we will develop methods for solving infinite-horizon nonlinear optimal control problems using the Legendre-Gauss (LG) and Legendre-Gauss-Radau (LGR) collocation points. Let N denote the number of collocation points, and let $P_N(\tau)$ be the N^{th}-degree Legendre polynomial. The LG, LGR, and LGL collocation points are respectively the roots of $P_N(\tau)$, the roots of $P_{N-1}(\tau) + P_N(\tau)$, and the roots of $\dot{P}_{N-1}(\tau)$ plus the points -1 and 1. A depiction of the LG and LGR collocation points is shown in Fig. 1 where it is seen that the LG points contain neither $\tau = -1$ nor $\tau = +1$, while the LGR points contain the point $\tau = -1$ but not the point $\tau = +1$. It is also seen that the LG points are symmetric about the origin whereas the LGR points are asymmetric.

3 Infinite-Horizon Optimal Control Problem

Consider the infinite-horizon optimal control problem

$$\min\ J = \int_0^\infty g(\mathbf{x}(t), \mathbf{u}(t))dt \quad \text{subject to} \quad \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{1}$$

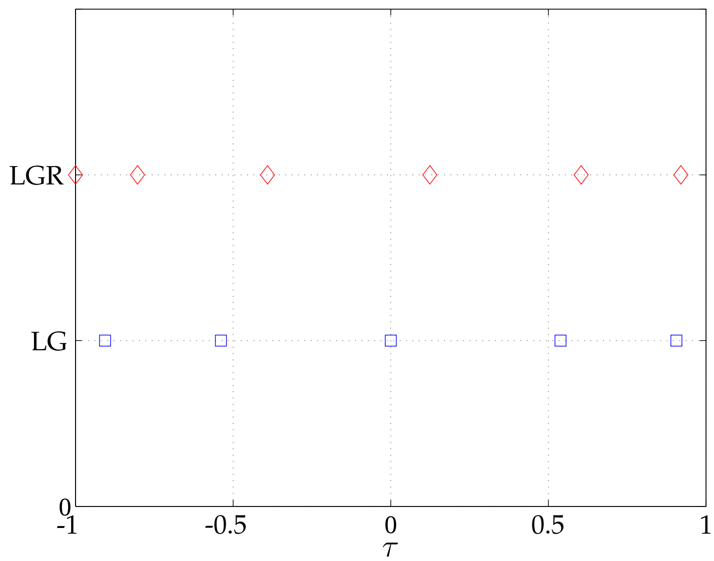


Figure 1: Schematic Showing LG and LGR Collocation Points.

where $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $\dot{\mathbf{x}}$ denotes the time derivative of $\mathbf{x}$. We make the change of variables $t = \phi(\tau)$ where ϕ is a differentiable, strictly monotonic function of τ that maps the half-open interval $[-1, 1)$ onto $[0, \infty)$. Three examples of such a function are

$$\phi_a(\tau) = \frac{1+\tau}{1-\tau}, \tag{2}$$

$$\phi_b(\tau) = \log\left(\frac{2}{1-\tau}\right), \tag{3}$$

$$\phi_c(\tau) = \log\left(\frac{4}{(1-\tau)^2}\right). \tag{4}$$

The change of variables $\phi_a(\tau)$ was originally proposed in Ref. 4, while the transformations $\phi_b(\tau)$ and $\phi_c(\tau)$ are introduced in this paper. These latter changes of variables produce slower growth in $t = \phi(\tau)$ as τ approaches $+1$, than that of $\phi_a(\tau)$. As we will see in the numerical experiments, better discretizations can be achieved by tuning the change of variables to the problem.

Define $T(\tau) = d\phi/d\tau \equiv \phi'(\tau)$. After changing variables from t to τ, the infinite-horizon optimal control problem becomes

$$\min \ J = \int_{-1}^{+1} T(\tau) g(\mathbf{x}(\tau), \mathbf{u}(\tau)) d\tau \quad \text{subject to} \quad \dot{\mathbf{x}}(\tau) = T(\tau)\mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau)), \quad \mathbf{x}(-1) = \mathbf{x}_0. \tag{5}$$

Here $\mathbf{x}(\tau)$ and $\mathbf{u}(\tau)$ denote the state and the control as a function of the new variable τ. Formally, the first-order optimality conditions for the finite horizon control problem Eq. (5), also called the Pontryagin minimum principle, is

$$\dot{\boldsymbol{\lambda}}(\tau) = -T(\tau)\nabla_x H(\mathbf{x}(\tau), \mathbf{u}(\tau), \boldsymbol{\lambda}(\tau)), \quad \boldsymbol{\lambda}(1) = \mathbf{0}, \tag{6}$$

$$\mathbf{0} = \nabla_u H(\mathbf{x}(\tau), \mathbf{u}(\tau), \boldsymbol{\lambda}(\tau)), \tag{7}$$

where $H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = g(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\mathsf{T}}\mathbf{f}(\mathbf{x}, \mathbf{u})$ is the Hamiltonian for Eq. (1).

4 Notation

Throughout the paper, we employ the following notation. First, we treat all vector functions of time as *row* vectors; that is, $\mathbf{x}(\tau) = [x_1(\tau), \; \cdots, \; x_n(\tau)] \in \mathbb{R}^n$, where n is the continuous-time dimension of $\mathbf{x}(\tau)$. $\mathbf{B}^{\mathsf{T}}$ denotes the transpose of a matrix $\mathbf{B}$. Given $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^n$, $\langle \mathbf{a}, \mathbf{b} \rangle$ is their dot product. If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, then $\nabla \mathbf{f}$ is the m by n Jacobian matrix whose i-th row is ∇f_i. In particular, the gradient of a scalar-valued function is a row vector. If $\phi : \mathbb{R}^{m \times n} \to \mathbb{R}$ and $\mathbf{X}$ is an m by n matrix, then $\nabla \phi$ denotes the m by n matrix whose (i, j) element is $(\nabla \phi(\mathbf{X}))_{ij} = \partial \phi(\mathbf{X}) / \partial X_{ij}$. If $\mathbf{A}$ is a matrix, then $\mathbf{A}_{i:j}$ is the submatrix formed by rows i through j, while $\mathbf{A}_i$ is the i-th row of $\mathbf{A}$. The Kronecker delta function is defined by $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

5 Pseudospectral Methods for Infinite-Horizon Optimal Control Problems

In this section we formulate discrete approximations to the nonlinear infinite-horizon optimal control problem described in Section 3. These discrete schemes are based on global collocation using either Gauss or Radau collocation points. As will be seen, these two schemes differ in their treatment of the horizon. For the Gauss quadrature scheme, the state at the horizon is recovered by quadrature after solving the discrete problem, while for the Radau scheme, the state at the horizon is a variable in the discrete scheme.

5.1 LG Collocation: Infinite-Horizon Gauss Pseudospectral Method

Consider the LG collocation points $(\tau_1, \ldots, \tau_N)$ on the interval $(-1, 1)$ and two additional *non-collocated* points $\tau_0 = -1$ (the initial time) and $\tau_{N+1} = 1$ (the terminal time, corresponding to $t = +\infty$). The state is approximated by a polynomial of degree at most N as

$$\mathbf{x}(\tau) \approx \sum_{j=0}^{N} \mathbf{X}_j L_j(\tau), \quad L_j(\tau) = \prod_{\substack{k=0 \\ k \neq j}}^{N} \frac{\tau - \tau_k}{\tau_j - \tau_k}, \qquad j = 0, \ldots, N, \tag{8}$$

where $\mathbf{X}_j \in \mathbb{R}^n$ and L_j is a basis of N^{th}-degree Lagrange polynomials. Notice that the basis includes the function L_0 corresponding to the initial time $\tau_0 = -1$, but not a function corresponding to $\tau_{N+1} = +1$. Differentiating the series of Eq. (8) and evaluating at the collocation point τ_i gives

$$\dot{\mathbf{x}}(\tau_i) \approx \sum_{j=0}^{N} \mathbf{X}_j \dot{L}_j(\tau_i) = \sum_{j=0}^{N} D_{ij} \mathbf{X}_j = \mathbf{D}_i \mathbf{X}, \tag{9}$$

where $\mathbf{D}_i$ is the i-th row of $\mathbf{D}$,

$$D_{ij} = \dot{L}_j(\tau_i), \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_0 \\ \vdots \\ \mathbf{X}_N \end{bmatrix}.$$

The rectangular $N \times (N+1)$ matrix $\mathbf{D}$ formed by the coefficients $D_{ij},\ (i = 1, \ldots, N;\ j = 0, \ldots, N)$ is the *Gauss Pseudospectral differentiation matrix* since it transforms the state approximations at $\tau_0, \ldots, \tau_N$ to the derivative of the state approximation at the collocation points $\tau_1, \ldots, \tau_N$.

Let $\mathbf{U}$ be an $N \times n$ matrix whose i-th row $\mathbf{U}_i$ is an approximation to the control $\mathbf{u}(\tau_i)$, $1 \leq i \leq N$. Our discrete approximation to the system dynamics $\dot{\mathbf{x}}(\tau) = T(\tau)\mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau))$ is obtained by evaluating the system dynamics at each collocation point and replacing $\dot{\mathbf{x}}(\tau_i)$ by its discrete approximation $\mathbf{D}_i\mathbf{X}$. Hence, the discrete approximation to the system dynamics is

$$\mathbf{D}_i \mathbf{X} = T_i \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i), \quad 1 \leq i \leq N, \tag{10}$$

where $T_i = T(\tau_i)$.

It is important to observe that the left-hand side of Eq. (10) contains approximations for the state at the initial point plus the LG points while the left-hand side contains approximations for the state (and control) at only the LG points. The objective function Eq. (5) is approximated by a Legendre-Gauss quadrature as

$$J = \int_{-1}^{+1} T(\tau) g(\mathbf{x}(\tau), \mathbf{u}(\tau)) d\tau \approx \sum_{i=1}^{N} w_i T_i g(\mathbf{X}_i, \mathbf{U}_i),$$

where w_i is the quadrature weight associated with τ_i. Thus the continuous-time nonlinear infinite-horizon optimal control problem of Eq. (5) is approximated by the following finite-dimensional NLP:

$$\min \sum_{i=1}^{N} w_i T_i g(\mathbf{X}_i, \mathbf{U}_i) \quad \text{subject to} \quad \mathbf{D}_i \mathbf{X} = T_i \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i), \quad 1 \le i \le N, \quad \mathbf{X}_0 = \mathbf{x}_0. \tag{11}$$

After solving this NLP, the state at the horizon can be estimated by quadrature as follows:

$$\mathbf{x}(+1) = \mathbf{x}(-1) + \int_{-1}^{+1} T(\tau) \mathbf{f}(\mathbf{x}(\tau), u(\tau)) d\tau \approx \mathbf{x}_0 + \sum_{i=1}^{N} w_i T_i \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i) = \mathbf{X}_{N+1}. \tag{12}$$

Although the change of variables $t = \phi(\tau)$ must have a singularity at $\tau = +1$, we never evaluate $T(\tau) = \phi'(\tau)$ at the singularity in Eq. (11) or Eq. (12), rather we evaluate T at the quadrature points which are all strictly less than 1.

The first-order optimality conditions for Eq. (11), also called as the KKT conditions, are obtained by differentiating the Lagrangian $\mathcal{L}$ with respect to the free components of $\mathbf{X}$ and $\mathbf{U}$. The Lagrangian associated with Eq. (11) is

$$\mathcal{L}(\mathbf{\Lambda}, \mathbf{X}, \mathbf{U}) = \sum_{i=1}^{N} \Big(w_i T_i g(\mathbf{X}_i, \mathbf{U}_i) + \langle \mathbf{\Lambda}_i, T_i \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i) - \mathbf{D}_i \mathbf{X} \rangle \Big), \tag{13}$$

where $\mathbf{\Lambda}$ is the N by n matrix of Lagrange multipliers. Differentiating the Lagrangian with respect to $\mathbf{X}_i$ and $\mathbf{U}_i$, $1 \le i \le N$, gives us the optimality conditions

$$\mathbf{D}_i^{\mathsf{T}}\mathbf{\Lambda} = T_i \nabla_x \left(w_i g(\mathbf{X}_i, \mathbf{U}_i) + \langle \mathbf{\Lambda}_i, \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i) \rangle \right), \tag{14}$$

$$\mathbf{0} = T_i \nabla_u \left(w_i g(\mathbf{X}_i, \mathbf{U}_i) + \langle \mathbf{\Lambda}_i, \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i) \rangle \right), \tag{15}$$

$1 \le i \le N$. Here $\mathbf{D}_i^{\mathsf{T}}$ is the i-th row of $\mathbf{D}^{\mathsf{T}}$. We note that the end point constraint of Eq. (12) could have been incorporated in the Lagrangian as the additional term

$$\left\langle \mathbf{\Lambda}_{N+1}, -\mathbf{X}_{N+1} + \sum_{i=1}^{N} w_i T_i \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i) \right\rangle.$$

When the Lagrangian is differentiated with respect to $\mathbf{X}_{N+1}$, however, we learn that the multiplier $\mathbf{\Lambda}_{N+1} = \mathbf{0}$. Consequently, the term corresponding to the end point constraint disappears from the problem.

5.1.1 LG Transformed Adjoint System

Analogous to Ref. 11, the first-order optimality conditions of the NLP can be reformed so that they become a discretization of the first-order optimality conditions for the continuous control problem Eq. (5). Let us define the following expressions:

$$\boldsymbol{\lambda} = \mathbf{W}^{-1}\mathbf{\Lambda} \quad \text{and} \qquad \mathbf{D}^{\dagger} = -\mathbf{W}^{-1}\mathbf{D}_{1:N}^{\mathsf{T}}\mathbf{W}, \tag{16}$$

were $\mathbf{W}$ is the diagonal matrix whose i-th diagonal element is w_i. Making these substitutions in Eqs. (14) and (15), we can rewrite the optimality conditions as

$$\mathbf{D}_i^{\dagger}\boldsymbol{\lambda} = -T_i \nabla_x H(\mathbf{X}_i, \mathbf{U}_i, \boldsymbol{\lambda}_i), \tag{17}$$

$$\mathbf{0} = \nabla_u H(\mathbf{X}_i, \mathbf{U}_i, \boldsymbol{\lambda}_i). \tag{18}$$

Hence, this transformation makes the discrete optimality conditions look very similar to the continuous Pontryagin minimum principle of Eqs. (6)–(7). There are two basic differences: The continuous derivative $\dot{\boldsymbol{\lambda}}(\tau)$ is replaced by the discrete analog $\mathbf{D}_i^{\dagger}\boldsymbol{\lambda}$ and boundary condition

$\boldsymbol{\lambda}(+1) = 0$ present in the minimum principle is missing from the discrete optimality conditions. We now explain how the boundary condition $\boldsymbol{\lambda}(+1) = \mathbf{0}$ is incorporated in Eq. (17).

Let $\mathcal{P}_N$ denote the space of polynomials of degree at most N. Furthermore, define the subspace

$$\mathcal{P}_N^1 = \{p \in \mathcal{P}_N : p(1) = 0\}.$$

For any given polynomial $p \in \mathcal{P}_N^1$, let the vector $\mathbf{p} \in \mathbb{R}^N$ be defined by $p_j = p(\tau_j)$, $1 \leq j \leq N$. In Theorem 1 of Ref. 9, we have shown that

$$\mathbf{D}_i^{\dagger}\mathbf{p} = \dot{p}(\tau_i), \quad 1 \leq i \leq N.$$

Therefore, the left side of Eq. (17) is the derivative at τ_i of the polynomial that vanishes at $\tau = +1$ and that passes through $\boldsymbol{\lambda}_j$ at $\tau = \tau_j$, $1 \leq j \leq N$. As a result, we have $\boldsymbol{\lambda}_{N+1} = \mathbf{0}$ where $\boldsymbol{\lambda}_{N+1}$ is the discrete costate approximation to $\boldsymbol{\lambda}(+1)$, the continuous costate at $\tau = +1$. For completeness, we note in an equivalent manner that, had the end point constraint of Eq. (12) been used in the formulation of the KKT conditions, the transformed adjoint system Eq. (16) would have included the condition $\boldsymbol{\lambda}_{N+1} = \boldsymbol{\Lambda}_{N+1} = \mathbf{0}$, again leading to $\boldsymbol{\lambda}_{N+1} = \mathbf{0}$.

The transformed conditions Eqs. (17)–(18) yield estimates for the costate at the collocation points τ_i. We can also estimate the costate at $\tau = -1$ by the expression

$$\boldsymbol{\lambda}_0 = -\mathbf{D}_0^{\mathsf{T}}\boldsymbol{\Lambda}.$$

The rationale for this estimate of the initial costate is based on equation (32) in Ref. 9 from which it follows that

$$\boldsymbol{\lambda}_0 = \sum_{j=1}^{N} w_j T_j \nabla_x H(\mathbf{X}_j, \mathbf{U}_j, \boldsymbol{\lambda}_j). \tag{19}$$

The continuous costate, on the other hand, satisfies

$$\boldsymbol{\lambda}(-1) = \boldsymbol{\lambda}(+1) - \int_{-1}^{+1} \dot{\boldsymbol{\lambda}}(\tau) d\tau = \int_{-1}^{+1} T(\tau)\nabla_x H(\mathbf{x}(\tau), \mathbf{u}(\tau), \boldsymbol{\lambda}(\tau)) d\tau. \tag{20}$$

Hence, the right side of Eq. (19) represents a quadrature approximation to the right side of Eq. (20). We refer to the LG collocation method developed in this section as the infinite-horizon version of the *Gauss pseudospectral method*.[5–7,9,10]

5.2 LGR Collocation: Infinite-Horizon Radau Pseudospectral Method

Consider the LGR collocation points $-1 = \tau_1 < \ldots < \tau_N < +1$, and the additional *noncollocated* point $\tau_{N+1} = 1$. The state is then approximated by a polynomial of degree at most N as

$$\mathbf{x}(\tau) \approx \sum_{j=1}^{N+1} \mathbf{X}_j L_j(\tau), \quad L_j(\tau) = \prod_{\substack{k=1 \\ k \neq j}}^{N+1} \frac{\tau - \tau_k}{\tau_j - \tau_k}, \qquad j = 1, \ldots, N+1, \tag{21}$$

where L_j is a basis of N^{th}-degree Lagrange polynomials. For the Radau scheme, the Lagrange interpolation points are $\tau_1 = -1$ through $\tau_{N+1} = +1$, while for the Gauss scheme, the interpolation points were $\tau_0 = -1$ through $\tau_N < +1$. Again, differentiating the series Eq. (21) and evaluating at the collocation point τ_i gives

$$\dot{\mathbf{x}}(\tau_i) \approx \sum_{j=1}^{N+1} \mathbf{X}_j \dot{L}_j(\tau_i) = \sum_{j=1}^{N+1} D_{ij} \mathbf{X}_j = \mathbf{D}_i \mathbf{X}, \quad i = 1, \ldots, N, \tag{22}$$

where

$$D_{ij} = \dot{L}_j(\tau_i) \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_{N+1} \end{bmatrix}.$$

Different from the LG scheme of the previous section, the state $\mathbf{X}_{N+1}$ at the horizon appears in the state discretization Eq. (22). The discrete approximation to the control problem is almost the same the LG scheme Eq. (11) except for the index on the initial condition $\mathbf{X}_1$ below corresponding to $\tau_1 = -1$:

$$\min \sum_{i=1}^{N} w_i T_i g(\mathbf{X}_i, \mathbf{U}_i) \quad \text{subject to} \quad \mathbf{D}_i \mathbf{X} = T_i \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i), \quad 1 \le i \le N, \quad \mathbf{X}_1 = \mathbf{x}_0. \tag{23}$$

As with the Gauss pseudospectral method, it is important to observe that the left-hand side of the collocation equations contains approximations for the state at the LGR points plus the terminal point $\tau = +1$, which corresponds to $t = +\infty$. The right-hand side of the collocation equations contains approximations of the state at only the LGR points. Moreover, because the state $\mathbf{X}$ in Eq. (23) contains an additional component $\mathbf{X}_{N+1}$ corresponding to $\tau_{N+1} = +1$, the state at the horizon is a variable in our Radau state discretization. Again, we point out that the singularity in $T(\tau)$ at $\tau = +1$ is avoided in Eq. (23) since we evaluate T at the collocation points τ_i, $1 \leq i \leq N$, where $\tau_N < 1$.

In order to better relate the Radau discretization to the continuous control problem, we utilize the Lagrangian

$$\mathcal{L}(\mathbf{\Lambda}, \mathbf{X}, \mathbf{U}) = \langle \mathbf{\Lambda_0}, \mathbf{x}_0 - \mathbf{X}_1 \rangle + \sum_{i=1}^{N} \Big(w_i T_i g(\mathbf{X}_i, \mathbf{U}_i) + \langle \mathbf{\Lambda}_i, T_i \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i) - \mathbf{D}_i \mathbf{X} \rangle \Big).$$

This differs from the Lagrangian Eq. (13) for the Gauss scheme by the term $\langle \mathbf{\Lambda_0}, \mathbf{x}_0 - \mathbf{X}_1 \rangle$ associated with the initial condition. Omitting this term from the Lagrangian leads to an asymmetry in the KKT conditions since otherwise $\mathbf{X}_1$ would a fixed parameter in the Lagrangian while $\mathbf{U}_1$ would be free. The optimality conditions, obtained by differentiating the Lagrangian with respect to the states $\mathbf{X}_1, \ldots, \mathbf{X}_{N+1}$ and the controls $\mathbf{U}_1, \ldots, \mathbf{U}_N$, are

$$\begin{aligned} \mathbf{D}_i^{\mathsf{T}} \mathbf{\Lambda} &= T_i \nabla_x \left(w_i g(\mathbf{X}_i, \mathbf{U}_i) + \langle \mathbf{\Lambda}_i, \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i) \rangle \right) - \delta_{1i} \mathbf{\Lambda_0}, \quad \mathbf{D}_{N+1}^{\mathsf{T}} \mathbf{\Lambda} = \mathbf{0}, && (24) \\ \mathbf{0} &= T_i \nabla_u \left(w_i g(\mathbf{X}_i, \mathbf{U}_i) + \langle \mathbf{\Lambda}_i, \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i) \rangle \right), && (25) \end{aligned}$$

where $1 \leq i \leq N$ and δ_{ij} denotes the Kronecker delta function; $\delta_{1i} = 1$ if $i = 1$ and $\delta_{1i} = 0$ otherwise. The $\mathbf{\Lambda_0}$ term only enters into the first equation corresponding to differentiation with respect to $\mathbf{X}_1$. The boundary condition $\mathbf{D}_{N+1}^{\mathsf{T}} \mathbf{\Lambda} = \mathbf{0}$ arises from differentiating the Lagrangian with respect to $\mathbf{X}_{N+1}$.

5.2.1 LGR Transformed Adjoint System

Again, we reformulate the the discrete optimality conditions of the NLP so that they resemble the first-order optimality conditions for the continuous control problem Eq. (5). Let us introduce the following expressions:

$$\boldsymbol{\lambda}_0 = \mathbf{\Lambda}_0, \quad \boldsymbol{\lambda} = \mathbf{W}^{-1} \mathbf{\Lambda}, \quad \boldsymbol{\lambda}_{N+1} = \mathbf{D}_{N+1}^{\mathsf{T}} \mathbf{\Lambda}, \quad \mathbf{D}^{\dagger} = -\mathbf{W}^{-1} \mathbf{D}_{1:N}^{\mathsf{T}} \mathbf{W} - \frac{1}{w_1} \mathbf{e}_1 \mathbf{e}_1^{\mathsf{T}}, \qquad (26)$$

where $\mathbf{e}_1$ is the first column of the identity matrix. Substituting Eq. (26) into the conditions of Eqs. (24)–(25) gives

$$\begin{aligned} \mathbf{D}_i^\dagger \boldsymbol{\lambda} &= -T_i \nabla_x H(\mathbf{X}_i, \mathbf{U}_i, \boldsymbol{\lambda}_i) + \frac{\delta_{1i}}{w_1}(\boldsymbol{\lambda_0} - \boldsymbol{\lambda_1}), \quad \boldsymbol{\lambda}_{N+1} = \mathbf{0}, &(27)\\ \mathbf{0} &= \nabla_u H(\mathbf{X}_i, \mathbf{U}_i, \boldsymbol{\lambda}_i). &(28) \end{aligned}$$

The $\boldsymbol{\lambda_1}$ term in Eq. (27) emerges from the $\mathbf{e}_1$ term in the definition of $\mathbf{D}^\dagger$. Moreover, from the definition of $\boldsymbol{\lambda}_{N+1}$, the following identity can be derived (see Ref. 8 for the details):

$$\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_{N+1} + \sum_{j=1}^{N} w_j T_j \nabla_x H(\mathbf{X}_i, \mathbf{U}_i, \boldsymbol{\lambda}_i). \tag{29}$$

Consequently, $\boldsymbol{\lambda}_0$ represents a quadrature approximation to the fundamental theorem of calculus Eq. (20). In Ref. 8, we also show that $\mathbf{D}^\dagger$ is a differentiation matrix for polynomials of degree $N-1$; more precisely, if p is a polynomial of degree at most $N-1$ with values $p_i = p(\tau_i)$, $1 \le i \le N$, then

$$(\mathbf{D}^\dagger \mathbf{p})_i = \dot{p}(\tau_i), \quad 1 \le i \le N.$$

Hence, except for the $\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_1$ term in the first equation of Eq. (27), this system represents a pseudospectral scheme for the costate equation based on polynomials of degree $N-1$. We refer to the LGR collocation method developed in this section as the infinite-horizon version of the *Radau pseudospectral method*.

5.2.2 Integrated Forms

In formulating the pseudospectral schemes, the left side of the state equations in Eqs. (11) and (23) contained the derivatives of Lagrange polynomials. As shown in Ref. 8 and 9, we can invert the nonsingular part of the differentiation matrix $\mathbf{D}$ to write the discrete dynamics in the form

$$\mathbf{X}_i = \mathbf{x}_0 + \sum_{j=1}^{N} A_{ij} T_j \mathbf{f}(\mathbf{X}_j, \mathbf{U}_j), \tag{30}$$

where $1 \leq i \leq N$ for LG collocation and $2 \leq i \leq N+1$ for LGR collocation. Here the matrix elements A_{ij} can be expressed as the integrals of Lagrange interpolating polynomials associated with the collocation points. More precisely, for LG collocation, we have

$$A_{ij} = \int_{-1}^{\tau_i} L_j^{\dagger}(\tau) d\tau, \quad L_j^{\dagger} = \prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{\tau - \tau_k}{\tau_j - \tau_k}, \quad j = 1, \ldots, N, \quad i = 1, \ldots, N. \tag{31}$$

For LGR collocation, the right side of Eq. (31) defines $A_{i+1,j}$. Computationally, the differential formulation of Eq. (10) of the system dynamics is more convenient since any nonlinear terms in $\mathbf{f}$ retain their sparsity in the discretization, while for the integrated version of Eq. (30), the nonlinear terms are nonsparse due to multiplication by the dense matrix $\mathbf{A}$.

6 Example

Consider the following nonlinear infinite-horizon optimal control problem. Minimize the cost functional

$$J = \frac{1}{2} \int_0^{\infty} \left(\log^2 y(t) + u(t)^2 \right) dt \tag{32}$$

subject to the dynamic constraint

$$\dot{y}(t) = y(t) \log y(t) + y(t) u(t) \tag{33}$$

with the initial condition

$$y(0) = 2. \tag{34}$$

The exact solution to this problem is

$$\begin{aligned} y^*(t) &= \exp(x^*(t)), \\ u^*(t) &= -(1+\sqrt{2}) x^*(t), \\ \lambda^*(t) &= (1+\sqrt{2}) \exp(-x^*(t)) x^*(t), \\ x^*(t) &= 2 \exp(-t\sqrt{2}). \end{aligned} \tag{35}$$

The example of Eqs. (32)–(34) was solved using the infinite-horizon Gauss and Radau pseudospectral methods described above using the three strictly monotonic transformations of the domain $\tau \in [-1,+1)]$ given in Eqs. (2)–(4). The solutions were obtained using the MATLAB open-source optimal control software GPOPS of Ref. 10 using the NLP solver SNOPT[12,13] with optimality and feasibility tolerances of 10^{-10} and 2×10^{-10}, respectively. In addition, the example was solved using the infinite-horizon method of Ref. 4 with all three transformations given in Eqs. (2)–(4). The solution was obtained for $N = 5$, 10, 15, 20, 25, and 30. The maximum base ten logarithm of the state, control, and costate errors are defined as

$$\begin{aligned} E_y &= \max_k \ \log_{10} \left|\mathbf{X}_k - y^*(\tau_k)\right| \\ E_u &= \max_k \ \log_{10} \left|\mathbf{U}_k - u^*(\tau_k)\right| \\ E_\lambda &= \max_k \ \log_{10} \left|\boldsymbol{\lambda}_k - \lambda_y^*(\tau_k)\right|. \end{aligned} \tag{36}$$

where in the case of the state and costate the index k spans the approximation points while in the case of the control k spans only the collocation points. We remind the reader that the state and costate are obtained on the *entire horizon* with the index $N+1$ corresponding to the state and costate at $\tau = +1$, or equivalently, at $t = +\infty$.

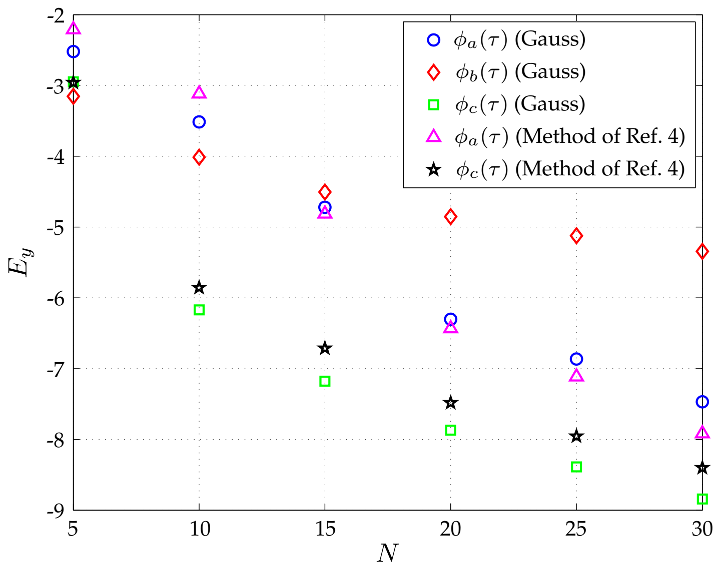
The errors obtained using the Gauss and Radau methods of this paper are shown, respectively, in Figs. 2 and 3 alongside the error obtained using the method of Ref. 4 with the transformation given in Eqs. (2) and (4). It is seen for all three transformations and for both methods of this paper, the state, control, and costate errors decrease in essentially a linear manner until $N = 30$, demonstrating an approximately exponential convergence rate. Furthermore, is it observed that either the Gauss or Radau method of this paper yields approximately the same error for a particular value of N and choice of transformation. Moreover, it is seen that the errors are largest and smallest, respectively, using the transformations of Eqs. (3) and (4). In fact, the transformation of Eq. (4) is at least one order of magnitude more accurate than either of the other two transformations. Finally, it is seen that the errors from the two methods of this paper using the transformation of Eq. (4) are significantly smaller than those obtained using the method of Ref. 4 (where the transformation of Eq. (2) are used). When the transformation of Eq. (4) is used, however, the state errors from the method of Ref. 4 are nearly the same as those obtained using the

Gauss and Radau methods, while the control and costate errors are approximately one order of magnitude larger using the method of Ref. 4. The results presented in this example demonstrate that the change of variables can be tuned to improve the accuracy of either the methods of this paper or the method of Ref. 4.

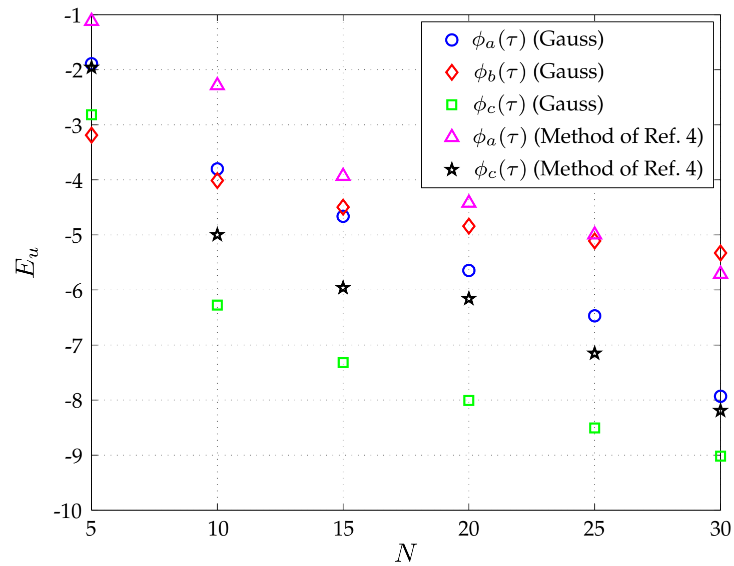
The different behavior of the functions given in Eqs. (2)–(4) is understood if we apply the change of variables to the continuous solution. The optimal state in the transformed coordinates is as follows:

$$\begin{aligned} y_{\mathrm{a}}(\tau) &= \exp\left(\exp\left(-2\sqrt{2}\left(\frac{1+\tau}{1-\tau}\right)\right)\right) \\ y_{\mathrm{b}}(\tau) &= \exp\left(\left(\frac{1-\tau}{2}\right)^{2\sqrt{2}}\right) \\ y_{\mathrm{c}}(\tau) &= \exp\left(\frac{(1-\tau)^{4\sqrt{2}}}{4^{2\sqrt{2}}}\right) \end{aligned}$$

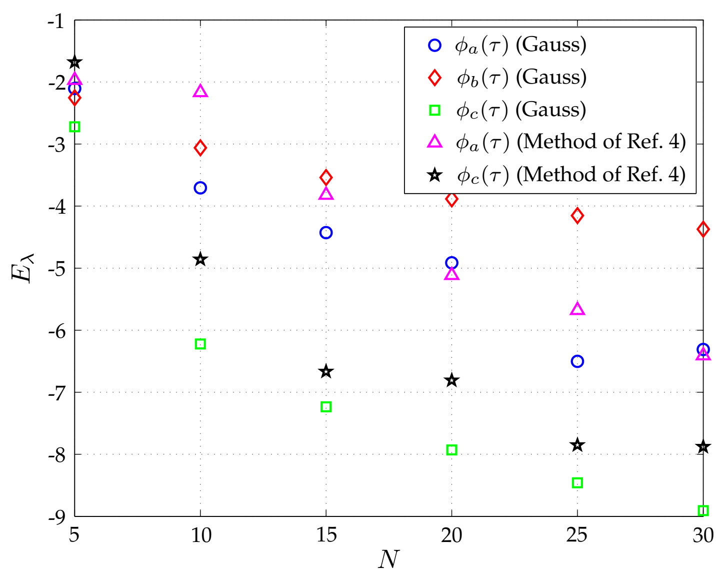
Here the subscripts a, b, and c correspond to the three choices of ϕ given in Eqs. (2)–(4). An advantage of using a logarithmic change of variables given in Eqs. (3) or (4), as compared to the function given in Eq. (2), is that logarithmic functions essentially move collocation points associated with large values of t to the left. Because the exact solution changes slowly when t is large, this leftward movement of the collocation points is beneficial since more collocation points are situated where the solution is changing most rapidly. The disadvantage of a logarithmic change of variables is seen in the function $\log(1-\tau)$ where the growth is so slow near $\tau = +1$ that the transformed solution possesses a singularity in a derivative at $\tau = +1$. In other words, the j-th derivative of a function of the form $(1-\tau)^{\alpha}$, where $\alpha > 0$ is not an integer, is singular at $\tau = +1$ for $j > \alpha$. In particular, $y_{\mathrm{b}}(\tau)$ has two derivatives at $\tau = +1$ but not three, while $y_{\mathrm{c}}(\tau)$ has five derivatives at $\tau = +1$ but not six. To achieve exponential convergence, $y(\tau)$ should be infinity smooth. For this particular problem, the choice Eq. (4) has the following nice properties: $y_{\mathrm{c}}(\tau)$ is relatively smooth with five derivatives, although not infinitely smooth, and collocation points corresponding to large values of t, where the solution changes slowly, are moved to the left [when compared to $t = (1+\tau)/(1-\tau)$] where the solution changes more rapidly. As a result, for $5 \le N \le 30$, the function of Eq. (4) yields a solution that is often two or more orders of magnitude more accurate than the other choices for ϕ.



(a) State Error.

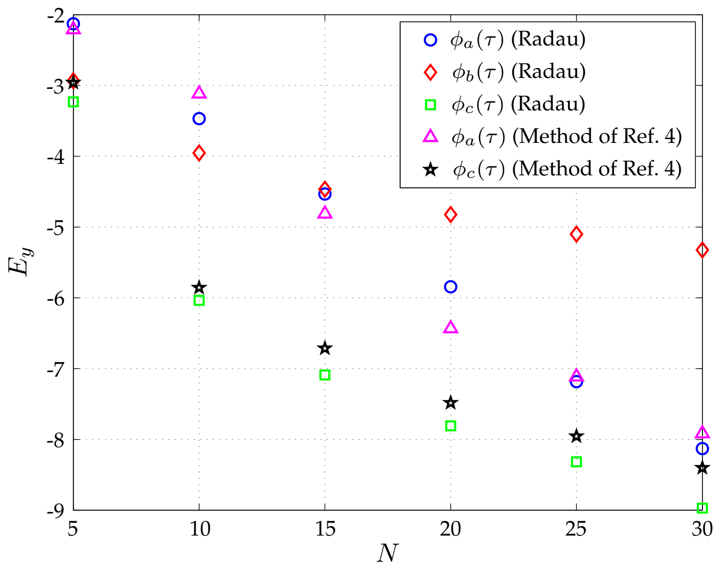


(b) Control Error.

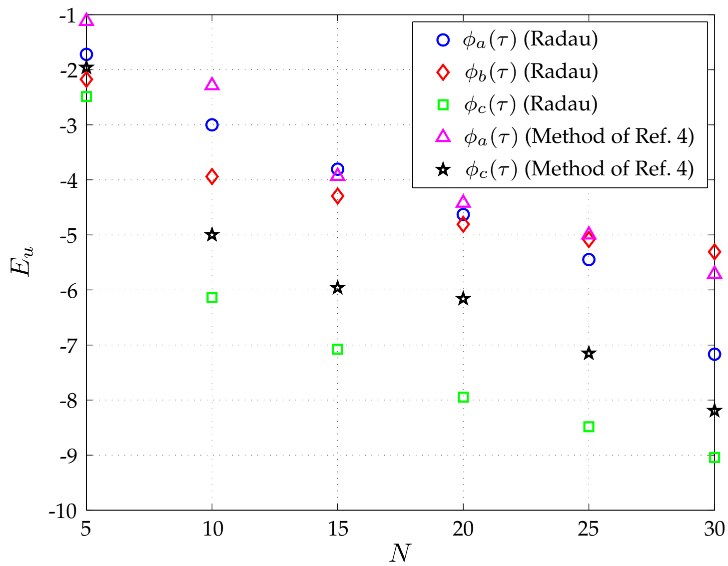


(c) Costate Error.

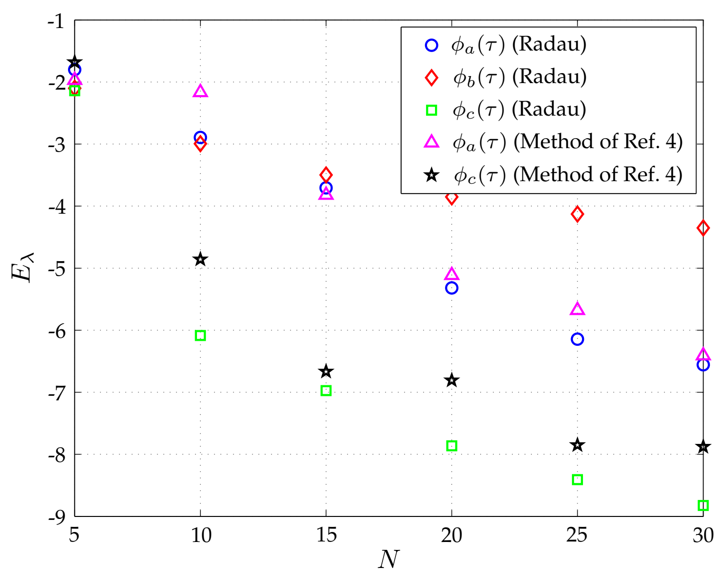
Figure 2: Maximum State, Control, and Costate Errors for Example Using Gauss Pseudospectral Method Alongside Errors Obtained Using the Method of Ref. 4 Using the Transformations $\phi_a(\tau)$ and $\phi_c(\tau)$ Given in Eqs. (2) and (4), Respectively.



(a) State Error.



(b) Control Error.



(c) Costate Error.

Figure 3: Maximum State, Control, and Costate Errors for Example Using Radau Pseudospectral Method Alongside Errors Obtained Using the Method of Ref. 4 Using the Transformations $\phi_a(\tau)$ and $\phi_c(\tau)$ Given in Eqs. (2) and (4), Respectively.

7 Comparison with Infinite-Horizon Method of Ref. 4

Ref. 4 also considers a pseudospectral approximation of the infinite horizon optimal control using LGR points. In Ref. 4, the time domain $\tau \in [-1, +1)$ is mapped to the domain $t \in [0, +\infty)$ using the particular change of variables $t = (1+\tau)/(1-\tau)$. Because this change of variables leads to a singularity in the transformed dynamics at $\tau = +1$, it is not possible to collocate at $\tau = +1$. The method of Ref. 4 avoids this singularity by *collocating and approximating* at only the Radau quadrature points for which $\tau_N < 1$. A fundamental difference between the pseudospectral method of Ref. 4 and the methods introduced in this paper lies in the manner in which the horizon is handled. In Ref. 4, the state is approximated by polynomials of degree $N-1$ using Lagrange polynomials associated with the N LGR points; the derivative of the state, a polynomial of degree $N-2$, is then collocated at the N LGR points. In this paper, the state is approximated using polynomials of degree N. The derivative of the state, a polynomial of degree $N-1$, is then collocated at either the N LG or LGR points. The extra dimension in the state variable is used to interpolate the state at the horizon. Hence, the state at the horizon is one of the variables in the NLP. This difference in the dimension of the state variable also results in a fundamental difference in the differentiation matrices. Our differentiation matrices are full rank. As a result, either of our pseudospectral schemes can be inverted to achieve an equivalent integrated form as explained in Section 5.2.2. On the other hand, the differentiation matrix of Ref. 4 is singular. Furthermore, we consider a general change of variables $t = \phi(\tau)$ and find that for a specific problem, better approximations to the continuous problem are achieved by using a function that grows more slowly than $\phi(\tau) = (1+\tau)/(1-\tau)$ near $\tau = +1$. Hence, by tuning the choice of ϕ to the problem, one can achieve a more accurate discretization.

8 Conclusions

Two pseudospectral methods have been presented for direct trajectory optimization and costate estimation for nonlinear infinite horizon optimal control problems using global collocation at Legendre-Gauss and Legendre-Gauss-Radau points. It was shown that the nonlinear programming problems which arise from a change of variables followed by either Gauss or Radau collocation includes an approximation to the state at $t = +\infty$. The Legendre-Gauss and Legendre-Gauss-Radau transformed adjoint systems connecting the KKT conditions of the nonlinear programming problem to the Pontryagin minimum principle were then derived. These transformed adjoint systems resulted in approximations for the costate at $t = +\infty$. Finally, it was shown that either of the methods developed in this paper can be written equivalently in either a differentiated or an integrated form. The results of this paper indicate that the use of Legendre-Gauss and Legendre-Gauss-Radau points lead to accurate approximations to a continuous nonlinear infinite-horizon optimal control problem in such a manner that the solution is obtained on the entire infinite-horizon. By tuning the change of variables used to map the infinite time domain $[0, \infty)$ to a finite interval, the accuracy in the discrete approximation was improved by several orders of magnitude.

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