Costate Approximation in Optimal Control Using Integral Gaussian Quadrature Orthogonal Collocation Methods

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Abstract

Two methods are presented for approximating the costate of optimal control problems in integral form using orthogonal collocation at Legendre-Gauss and Legendre-Gauss-Radau points. It is shown that the derivative of the costate of the continuous-time optimal control problem is equal to the negative of the costate of the integral form of the continuous-time optimal control problem. Using this continuous-time relationship between the differential and integral costate, it is shown that the discrete approximations of the differential costate using Legendre-Gauss and Legendre-Gauss-Radau collocation are related to the corresponding discrete approximations of the integral costate via integration matrices. The approach developed in this paper provides a way to approximate the costate of the original optimal control problem using the Lagrange multipliers of the integral form of the Legendre-Gauss and Legendre-Gauss-Radau collocation methods. The methods are demonstrated on two examples where it is shown that both the differential and integral costate converge exponentially as a function of the number of Legendre-Gauss or Legendre-Gauss-Radau points.

1 Introduction

Over the past two decades, direct collocation methods have become popular in the numerical solution of nonlinear optimal control problems. In a direct collocation method, the state and control are discretized at a set of appropriately chosen points in the time interval of interest. The continuous-time optimal control problem is then transcribed to a finite-dimensional nonlinear programming problem (NLP) and the NLP is solved using well known software.[1,2] Recently, a great deal of research as been done on the class of *Gaussian quadrature orthogonal collocation methods*.[3–25] In a Gaussian quadrature orthogonal collocation method, the state is approximated using a basis of either Lagrange or Chebyshev polynomials, and the dynamics are collocated at points associated with a Gaussian quadrature. The most common Gaussian quadrature collocation points are Legendre-Gauss (LG),[3,4,6–9,11] Legendre-Gauss-Radau (LGR),[9,10,10–14] and Legendre- Gauss-Lobatto (LGL).[15–25] All three types of Gaussian quadrature points are defined on the domain $[-1, 1]$, but differ in that the LG points include neither of the endpoints, the LGR points include one of the endpoints, and the LGL points include both of the endpoints. The use of global polynomials together with Gaussian quadrature collocation points is known to provide accurate approximations that converge exponentially fast for problems whose solutions are smooth.[6,9–11,14] An advantage of these methods is that by computing the solution of the control problem accurately at a small number of carefully chosen points, one obtains an accurate global

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approximation. Since the problem solution is approximated in a small dimensional space, the numerical algorithms can be very efficient.

The costate variable in an optimal control problem is related to the sensitivity of the objective function to perturbations in the system dynamics. The costate is essentially the derivative of the objective function with respect to a perturbation in the system dynamics. For example, see Ref. 26 and the references therein. Previous research on costate approximation using Gaussian quadrature collocation has focused on the use of the differential form of the collocation methods. However, recent research strongly indicates that their may be computational advantages to using the integral form of LG and LGR collocation over the differential form. In fact, the most current implementation of LGR collocation is the MATLAB optimal control software $\mathbb{GPOPS}-\mathbb{II}$[27] which uses the integral form of LGR collocation by default because it has been found through a variety of examples that the integral form provides more consistent results. The implicit integral forms of LG and LGR collocation are consistent with the implementations used by established optimal control software packages such as *SOCS*,[28] *DIRCOL*,[29] *OTIS*,[30] *ICLOCS*,[31] and *ACADO*.[32]

It is important to note that while the differential and integral forms of LG and LGR collocation produce equivalent primal solutions (that is, state and control), these two formulations produce completely different dual variables. Moreover, the discretized versions of the integral and differential dynamics have much different numerical characteristics. For example, when refining the mesh in order to achieve a specified error tolerance, the error estimates for the integral dynamics are much more stable and reliable than the error estimates derived from the differential dynamics.[27,33] Based on the computational importance of the integral form of the collocation methods, this paper will analyze the relationship between the Lagrange multipliers associated with the discretized integral forms, and the costate of the continuous optimal control problem. Our earlier work[9–11] analyzed the relationships between the Lagrange multipliers arising in the discrete and continuous differential formulations of the optimal control problem.

The approach developed in this paper provides a way to approximate the costate of the original optimal control problem using the Lagrange multipliers of the integral form of the LG and LGR collocation methods. Transformations are derived that relate the Lagrange multipliers of the integral forms of the LG and LGR collocation methods to the costate of the original optimal control problem. These transformations are derived by writing the original continuous-time optimal control problem in integral form. A new continuous-time dual variable called the *integral costate* is then introduced, where the integral costate is the Lagrange multiplier of the integral dynamic constraint. The first-order optimality conditions of the integral form of the optimal control problem are derived in terms of the integral costate. The integral form of the optimal control problem is then discretized using the integral LG and LGR collocation methods and relationships between the discrete form of the integral costate and the costate of the original differential optimal control problem are developed. It is shown that the Legendre-Gauss-Radau integration matrix that relates the differential costate to the integral costate is singular while the corresponding Legendre-Gauss integration matrix is full rank. These relationships lead to a way to approximate the costate of the original optimal control problem using the Lagrange multipliers of the integral form of the LG and LGR collocation methods. The two methods developed in this paper are demonstrated on two examples where it is found that the costate converges exponentially, consistent with the analysis in Ref. 34 for unconstrained control problems with smooth solutions. Although we focus on unconstrained control problems in this paper, the relations we establish between the continuous costate and the Lagrange multipliers for the discrete integral forms are also applicable for problems with control constraints or endpoint constraints. On the other hand, when state constraints are present, the relationship between the continuous costate and the multipliers in the discrete problem is more complex as shown in Refs. 35 and 36.

This paper is organized as follows. In Section 2 we introduce the conventions and notation used in the remainder of this paper. In Section 3 we formulate the continuous-time optimal control problem with the dynamic constraints formulated in both differential and integral form, and we present the first-order optimality conditions for each form. In Sections 4 and 5, we present the Legendre-Gauss and Legendre-Gauss-Radau collocation methods in both differential and integral forms, derive the first-order optimality conditions in each form, develop the transformed adjoint system, and derive a costate approximation in terms of the Lagrange multipliers of the integral forms. In Section 6 we provide two examples that demonstrate the accuracy of the LG and LGR costate approximation methods derived in this paper. Finally, in Section 8 we provide conclusions on our work.

2 Conventions and Notation

The following notation and conventions are used throughout this paper. Except where explicitly noted, vectors in this paper are row vector. In particular, if $\mathbf{y}(\tau) \in \mathbb{R}^n$ is the state vector at time τ, then $\mathbf{y}(\tau) = [y_1(\tau), \cdots, y_n(\tau)]$. Generally, if $\mathbf{Y}$ is a matrix, then $\mathbf{Y}_i$ is the i-th row of $\mathbf{Y}$, while $\mathbf{Y}_{i:j}$ denotes the submatrix formed by rows i through j. Two exceptions are the differentiation matrix $\mathbf{D}$ and the integration matrix $\mathbf{A}$, in which case $\mathbf{D}_i$ and $\mathbf{A}_i$ refers to the i^{th} *column* of $\mathbf{D}$ or $\mathbf{A}$. Finally, $\mathbf{D}^\top$ denotes the transpose of matrix $\mathbf{D}$, and $\mathbf{D}_i^\top$ denotes the transpose of the i^{th} column of $\mathbf{D}$. Given vectors $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$, the notation $\langle \mathbf{x}, \mathbf{y} \rangle$ is used to denote the standard Euclidean inner product between $\mathbf{x}$ and $\mathbf{y}$. Furthermore, if $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\mathbf{Y}$ is N by n, then $\mathbf{f}(\mathbf{Y})$ is the matrix whose i-th row is $\mathbf{f}(\mathbf{Y}_i)$. If $\mathbf{y} \in \mathbb{R}^n$, then $\nabla \mathbf{f}(\mathbf{y})$ denotes the Jacobian of $\mathbf{f}$ evaluate at $\mathbf{y}$; the Jacobian is an $m \times n$ matrix whose i-th row is $\nabla \mathbf{f}_i(\mathbf{y})$. In particular, the gradient of a scalar-valued function is a row vector. Finally, the Kronecker delta function is defined by $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

3 Continuous-Time Bolza Optimal Control Problem

In this section we state the differential and integral forms of the continuous-time Bolza optimal control problem under consideration in this paper. In addition, we provide the first-order optimality conditions of each form of the problem and explain how these two sets of optimality conditions are related to one another.

3.1 Differential and Integral Forms of Optimal Control Problem

Consider the following continuous-time optimal control problem defined on the interval $\tau \in [-1, +1]$. Determine the state $\mathbf{y}(\tau) \in \mathbb{R}^n$ and the control $\mathbf{u}(\tau) \in \mathbb{R}^m$ that minimize the cost functional

$$J = \Phi(\mathbf{y}(+1)) + \int_{-1}^{+1} g(\mathbf{y}(\tau), \mathbf{u}(\tau)) d\tau, \tag{1}$$

subject to the dynamic constraint

$$\dot{\mathbf{y}}(\tau) - \mathbf{f}(\mathbf{y}(\tau), \mathbf{u}(\tau)) = \mathbf{0}, \quad \tau \in [-1, +1], \tag{2}$$

and the boundary condition

$$\mathbf{y}(-1) = \mathbf{y}_0. \tag{3}$$

It is noted that the time interval $\tau \in [-1, +1]$ can be transformed to the interval $[t_0, t_f]$ via the affine transformation

$$t = \frac{t_f - t_0}{2}\tau + \frac{t_f + t_0}{2}.$$

Henceforth, Eqs. (1)–(3) will be referred to as the *differential optimal control problem*.

The differential optimal control problem given in Eqs. (1)–(3) can be re-written in the following integral form. In particular, integrating the dynamics given in Eq. (2), we have

$$\mathbf{y}(\tau) = \mathbf{y}(-1) + \int_{-1}^{\tau} \mathbf{f}(\mathbf{y}(\tau), \mathbf{u}(\tau)) \, d\tau.$$

The optimal control problem in integral form is then stated as follows. Determine the state $\mathbf{y}(\tau) \in \mathbb{R}^n$ and the control $\mathbf{u}(\tau) \in \mathbb{R}^m$ that minimize the cost functional

$$J = \Phi(\mathbf{y}(+1)) + \int_{-1}^{+1} g(\mathbf{y}(\tau), \mathbf{u}(\tau)) d\tau, \tag{4}$$

subject to the integral constraint

$$\mathbf{y}(\tau) - \mathbf{y}(-1) - \int_{-1}^{\tau} \mathbf{f}(\mathbf{y}(\tau), \mathbf{u}(\tau)) \, dt = \mathbf{0}, \quad \tau \in [-1, +1], \tag{5}$$

and the boundary condition

$$\mathbf{y}(-1) = \mathbf{y}_0. \tag{6}$$

Henceforth, Eqs. (4)–(6) will be referred to as the *integral optimal control problem*.

3.2 First-Order Optimality Conditions of Differential and Integral Forms

The first-order optimality conditions for the differential optimal control problem, given by the Pontryagin minimum principle, are[37]

$$\begin{aligned}
\dot{\mathbf{y}}(\tau) &= \mathbf{f}(\mathbf{y}(\tau), \mathbf{u}(\tau)), \quad \tau \in [-1, +1], & (7)\\
\mathbf{y}(-1) &= \mathbf{y}_0, & (8)\\
\mathbf{0} &= \nabla_u \mathcal{H}(\mathbf{y}(\tau), \mathbf{u}(\tau), \boldsymbol{\lambda}(\tau)), \quad \tau \in [-1, +1], & (9)\\
-\dot{\boldsymbol{\lambda}}(\tau) &= \nabla_y \mathcal{H}(\mathbf{y}(\tau), \mathbf{u}(\tau), \boldsymbol{\lambda}(\tau)), \quad \tau \in [-1, +1] & (10)\\
\boldsymbol{\lambda}(+1) &= \nabla \Phi(\mathbf{y}(+1)). & (11)
\end{aligned}$$

Here $\mathcal{H}$ is the Hamiltonian defined by

$$\mathcal{H}(\mathbf{y}, \mathbf{u}, \boldsymbol{\lambda}) = g(\mathbf{y}, \mathbf{u}) + \langle \boldsymbol{\lambda}, \mathbf{f}(\mathbf{y}, \mathbf{u}) \rangle, \tag{12}$$

and $\boldsymbol{\lambda}$ is the Lagrange multiplier associated with the differential dynamics given in Eq. (7).

The first-order optimality conditions for the integral optimal control problem, derived in the Appendix, are

$$\begin{aligned} \mathbf{y}(\tau) &= \mathbf{y}(-1) + \int_{-1}^{\tau} \mathbf{f}(\mathbf{y}(t), \mathbf{u}(t)) dt, \quad \tau \in [-1, +1], & (13) \\ \mathbf{y}(-1) &= \mathbf{y}_0, & (14) \\ \mathbf{0} &= \nabla_u \mathcal{H}(\mathbf{y}(\tau), \mathbf{u}(\tau), \boldsymbol{\lambda}(\tau)), \quad \tau \in [-1, +1], & (15) \\ \mathbf{r}(\tau) &= \nabla_y \mathcal{H}(\mathbf{y}(\tau), \mathbf{u}(\tau), \boldsymbol{\lambda}(\tau)), \quad \tau \in [-1, +1], & (16) \end{aligned}$$

where

$$\boldsymbol{\lambda}(\tau) = \nabla \Phi(\mathbf{y}(+1)) + \int_{\tau}^{+1} \mathbf{r}(t) \; dt, \tag{17}$$

and $\mathbf{r}$ is the multiplier associated with the integral dynamics of Eq. (13). Thus, Eq. (17) gives the relationship between the multipliers in the differential and integral formulations. Differentiating Eq. (17), we see that $\mathbf{r}(\tau) = -\dot{\boldsymbol{\lambda}}(\tau)$; that is, the multiplier for the integral dynamics is the negative derivative of the multiplier for the differential dynamics. The remainder of this paper is devoted to deriving two discrete approximations of the differential costate, $\boldsymbol{\lambda}(\tau)$, using discrete approximations of the integral costate, $\mathbf{r}(\tau)$.

4 Costate Approximation Using Integral Legendre-Gauss Collocation

In this section we present the LG collocation method, and establish the relation between the integral and differential discretized problems.

4.1 Differential Form of Legendre-Gauss Collocation

The differential optimal control problem is now approximated using collocation at Legendre-Gauss (LG) points (see Refs. 6, 9–11). The LG points are denoted $(\tau_1, \ldots, \tau_N)$ and are defined on the open interval $(-1, +1)$. The state is approximated by the polynomial

$$\mathbf{y}(\tau) \approx \mathbf{Y}(\tau) = \sum_{i=0}^{N} \mathbf{Y}_i L_i(\tau), \quad L_i(\tau) = \prod_{\substack{j=0 \\ j \neq i}}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \tag{18}$$

where $\tau_0 = -1$ is an additional point where we approximate the state, and $L_i(\tau)$, $i = 0, \ldots, N$, is a basis of Lagrange polynomials of degree N with support points $(\tau_0, \ldots, \tau_N)$. The time derivative of the state approximation at $\tau = \tau_i$, $1 \leq i \leq N$, is

$$\dot{\mathbf{y}}(\tau_i) \approx \dot{\mathbf{Y}}(\tau_i) = \sum_{j=0}^{N} \mathbf{Y}_j \dot{L}_j(\tau_i) = \sum_{j=0}^{N} \mathbf{Y}_j D_{ij} = [\mathbf{D}\mathbf{Y}_{0:N}]_i \,, \tag{19}$$

where $\mathbf{Y}_j = \mathbf{Y}(\tau_j)$ and $\mathbf{D}$ is the $N \times (N+1)$ *Legendre-Gauss (LG) differentiation matrix* whose elements are given by $D_{ij} = \dot{L}_j(\tau_i)$. Note that we only collocate the dynamics at the quadrature points τ_i, $1 \le i \le N$, not at the initial time $\tau_0 = -1$. If $\mathbf{w} = (w_1, \ldots, w_N)$ is the row vector of LG quadrature weights and $\tau_{N+1} = +1$ is the terminal time, then the discretized control problem is

$$\min \quad J = \Phi(\mathbf{Y}_{N+1}) + \sum_{j=1}^{N} w_j g(\mathbf{Y}_j, \mathbf{U}_j), \tag{20}$$

subject to the collocated dynamics

$$\begin{aligned} \mathbf{D}\mathbf{Y}_{0:N} - \mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}) &= \mathbf{0}, & (21) \\ \mathbf{Y}_{N+1} - \mathbf{Y}_0 - \mathbf{w}\mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}) &= \mathbf{0}, & (22) \\ \mathbf{Y}_0 &= \mathbf{y}_0, & (23) \end{aligned}$$

It is noted for LG collocation that Eq. (22) provides an LG quadrature approximation, $\mathbf{Y}_{N+1}$, of the state at the final noncollocated point $\tau_{N+1} = +1$. The NLP described by Eqs. (20)–(23) will be referred to as the *differential Legendre-Gauss (LG) collocation method*.

4.2 KKT Conditions Using Differential Legendre-Gauss Collocation

In Ref. 10, it is shown that the KKT conditions for the differential LG collocation method associated with Eqs. (20)–(23) can be written in the following form:

$$\mathbf{D}\mathbf{Y}_{0:N} = \mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}), \quad \mathbf{Y}_0 = \mathbf{y}_0, \tag{24}$$

$$\mathbf{Y}_{N+1} = \mathbf{Y}_0 + \mathbf{w}\mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}), \tag{25}$$

$$\mathbf{0} = \nabla_u \mathcal{H}(\mathbf{Y}_i, \mathbf{U}_i, \boldsymbol{\lambda}_i), \tag{26}$$

$$(\mathbf{D}^\dagger \boldsymbol{\lambda}_{1:N+1})_i = -\nabla_y \mathcal{H}(\mathbf{Y}_i, \mathbf{U}_i, \boldsymbol{\lambda}_i), \tag{27}$$

$$\boldsymbol{\lambda}_{N+1} = \nabla \Phi(\mathbf{Y}_{N+1}), \tag{28}$$

$i = 1, 2, \ldots, N$, where $\mathbf{D}^\dagger$ is defined by

$$D^\dagger_{ij} = -\frac{w_j}{w_i} D_{ji}, \quad 1 \le i, j \le N, \quad \text{and} \quad D^\dagger_{i,N+1} = -\sum_{j=1}^{N} D^\dagger_{ij}. \tag{29}$$

It was shown in Theorem 1 of Ref. 9, that $\mathbf{D}^\dagger$ is a differentiation matrix for the space of polynomials of degree N. More precisely, if b is a polynomial of degree at most N and $\mathbf{b} \in \mathbb{R}^{N+1}$ is the vector whose i^{th} element is $b_i = b(\tau_i)$, $1 \le i \le N+1$, then $(\mathbf{D}^\dagger \mathbf{b})_i = \dot{b}(\tau_i)$. In the transcription of Eqs. (24)–(28), the state is differentiated by a matrix $\mathbf{D}$ [given by Eq. (19)] which is based on the derivatives of polynomials of degree N with coefficients at the N LG points plus the initial noncollocated point $\tau_0 = -1$, whereas the costate is differentiated by a matrix $\mathbf{D}^\dagger$ [given by Eq. (29)] which is based on the derivatives of polynomials of degree N with coefficients at the N LG points plus the terminal noncollocated point $\tau_{N+1} = +1$.

4.3 Integral Form of Legendre-Gauss Collocation

The integral optimal control problem is now discretized using the integral form of LG collocation. It has been shown in Ref. 10 that the LG differentiation matrix $\mathbf{D}$ given by Eq. (19) has the property that the square matrix $\mathbf{D}_{1:N}$ obtained by removing the first column of $\mathbf{D}$ is full-rank and $-\mathbf{D}_{1:N}^{-1}\mathbf{D}_0 = \mathbf{1}$. We multiply the differential dynamics in Eq. (24) by $\mathbf{A} = \mathbf{D}_{1:N}^{-1}$ to obtain

$$\mathbf{Y}_{1:N} = \mathbf{1}\mathbf{Y}_0 + \mathbf{A}\mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}), \tag{30}$$

where $\mathbf{1}$ is an $N \times 1$ column vector of all ones. Combining Eq. (30) with (25) gives the discretized dynamics for the integral formulation. The integral optimal control problem of Eqs. (4)–(6) can then be approximated via the following finite-dimensional nonlinear programming problem: Minimize the cost function of Eq. (20) subject to the algebraic constraints

$$\mathbf{Y}_{1:N} = \mathbf{1}\mathbf{y}_0 + \mathbf{A}\mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}), \tag{31}$$

$$\mathbf{Y}_{N+1} = \mathbf{y}_0 + \mathbf{w}\mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}). \tag{32}$$

The NLP described by the objective function of Eq. (20) and the dynamics of Eqs. (31) and (32) will be referred to as the *integral Legendre-Gauss (LG) collocation method*.

4.4 KKT Conditions Using Integral Legendre-Gauss Collocation

The KKT conditions of the integral LG collocation method are found by taking the partial derivatives of the Lagrangian $\mathcal{L}$ of the NLP with respect to every free variable and setting the result to zero. The Lagrangian is

$$\begin{aligned}\mathcal{L} \;=\; & \Phi(\mathbf{Y}_{N+1}) + \sum_{i=1}^{N} w_i g(\mathbf{Y}_i, \mathbf{U}_i) + \langle \mathbf{R}_{1:N}, \mathbf{1}\mathbf{y}_0 + \mathbf{A}\mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}) - \mathbf{Y}_{1:N} \rangle \\ & + \langle \mathbf{R}_{N+1}, \mathbf{y}_0 + \mathbf{w}\mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}) - \mathbf{Y}_{N+1} \rangle,\end{aligned}$$

where $\mathbf{R}$ is the $N+1$ by n matrix of multipliers associated with the discrete dynamics. The partial derivatives with respect to the control and the state yield

$$\begin{aligned}\mathbf{0} &= w_i \nabla_u g(\mathbf{Y}_i, \mathbf{U}_i) + \nabla_u \langle (\mathbf{A}^{\mathsf{T}}\mathbf{R}_{1:N})_i, \mathbf{f}(\mathbf{Y}_i, \mathbf{U}_i) \rangle + w_i \nabla_u \langle \mathbf{R}_{N+1}, \mathbf{f}(\mathbf{Y}_i, \mathbf{U}_i) \rangle, \quad 1 \le i \le N, \\ \mathbf{R}_i &= w_i \nabla_y g(\mathbf{Y}_i, \mathbf{U}_i) + \nabla_y \langle (\mathbf{A}^{\mathsf{T}}\mathbf{R}_{1:N})_i, \mathbf{f}(\mathbf{Y}_i, \mathbf{U}_i) \rangle + w_i \nabla_y \langle \mathbf{R}_{N+1}, \mathbf{f}(\mathbf{Y}_i, \mathbf{U}_i) \rangle, \quad 1 \le i \le N, \\ \mathbf{R}_{N+1} &= \nabla \Phi(\mathbf{Y}_{N+1}).\end{aligned}$$

We make the change of variables

$$\mathbf{r}_i = \mathbf{R}_i / w_i, \quad 1 \le i \le N, \quad \mathbf{r}_{N+1} = \mathbf{R}_{N+1}, \quad A_{ji} = \frac{w_i}{w_j} A_{ij}^{\dagger}. \tag{33}$$

After these substitutions and after dividing the i^{th} equation by w_i, we obtain

$$\begin{aligned} \mathbf{0} &= \nabla_u \mathcal{H}(\mathbf{Y}_i, \mathbf{U}_i, \boldsymbol{\mu}_i), &(34)\\ \mathbf{r}_i &= \nabla_y \mathcal{H}(\mathbf{Y}_i, \mathbf{U}_i, \boldsymbol{\mu}_i), &(35)\\ \mathbf{r}_{N+1} &= \nabla \Phi(\mathbf{Y}_{N+1}), &(36) \end{aligned}$$

$i = 1, 2, \ldots, N$, where

$$\boldsymbol{\mu}_i = \mathbf{r}_{N+1} + (\mathbf{A}^\dagger \mathbf{r})_i. \tag{37}$$

Because $\boldsymbol{\lambda}_{N+1} = \mathbf{r}_{N+1}$ by Eq. (28), we see that Eqs. (34)–(35) and Eqs. (26)–(27) are equivalent if the following conditions hold:

$$\text{(a)} \;\; \boldsymbol{\lambda}_{1:N} = \mathbf{1}\mathbf{r}_{N+1} + \mathbf{A}^\dagger \mathbf{r}_{1:N} \quad \text{and} \quad \text{(b)} \;\; \mathbf{r}_{1:N} = -\mathbf{D}^\dagger \boldsymbol{\lambda}_{1:N+1}.$$

We will now show that conditions (a) and (b) are equivalent. This equivalence is based on the following key property: $(\mathbf{D}^\dagger_{1:N})^{-1} = -\mathbf{A}^\dagger$. For example, if (b) holds, then by the definition of $\mathbf{D}^\dagger_{N+1}$ in Eq. (29), we have

$$-\mathbf{r}_{1:N} = \mathbf{D}^\dagger \boldsymbol{\lambda}_{1:N+1} = \mathbf{D}^\dagger_{1:N} \boldsymbol{\lambda}_{1:N} + \mathbf{D}^\dagger_{N+1} \boldsymbol{\lambda}_{N+1} = \mathbf{D}^\dagger_{1:N} \boldsymbol{\lambda}_{1:N} - \mathbf{D}^\dagger_{1:N} \mathbf{1} \boldsymbol{\lambda}_{N+1}.$$

We multiply by $(\mathbf{D}^\dagger_{1:N})^{-1} = -\mathbf{A}^\dagger$ to obtain (a). The identity $(\mathbf{D}^\dagger_{1:N})^{-1} = -\mathbf{A}^\dagger$ is now established.

Theorem 1. *The matrix $\mathbf{A}^\dagger$ defined in Eq. (33) is a backwards integration matrix for the space of polynomials of degree $N-1$. That is, if p is a polynomial of degree at most $N-1$ and $\mathbf{p} \in \mathbb{R}^N$ is the vector with i^{th} component $p_i = p(\tau_i)$, then*

$$(\mathbf{A}^\dagger \mathbf{p})_i = \int_{\tau_i}^{+1} p(t) dt. \tag{38}$$

Moreover, $-\mathbf{A}^\dagger = (\mathbf{D}^\dagger_{1:N})^{-1}$.

Proof. Let p and q denote polynomials of degree at most $N-1$ such that $p_j = p(\tau_j)$ and $q_j = q(\tau_j)$ for $j = 1, \ldots, N$. Changing the order of integration, we have

$$\int_{-1}^{+1} \left[q(\tau) \int_{-1}^{\tau} p(t)\, dt \right] d\tau = \int_{-1}^{+1} \left[p(\tau) \int_{\tau}^{+1} q(t)\, dt \right] d\tau. \tag{39}$$

Since p and q are polynomials of degree at most $N-1$, it follows that

$$p(\tau) \int_{\tau}^{+1} q(t)\, dt \quad \text{and} \quad q(\tau) \int_{-1}^{\tau} p(t)\, dt$$

are polynomials of degree at most $2N-1$. Since LG quadrature is exact for polynomials of degree at most $2N-1$, the integrals in Eq. (39) can be replaced by their LG quadrature equivalents to obtain

$$\sum_{j=1}^{N} w_j q_j \int_{-1}^{\tau_j} p(t)\, dt = \sum_{i=1}^{N} w_i p_i \int_{\tau_i}^{+1} q(t)\, dt. \tag{40}$$

In Ref. 10 it is shown that

$$\int_{-1}^{\tau_j} p(t)dt = (\mathbf{A}\mathbf{p})_j, \quad 1 \le j \le N, \tag{41}$$

where $\mathbf{p}$ is a column vector. Let $L_j^\dagger$ denote the Lagrange basis functions defined by

$$L_j^\dagger = \prod_{\substack{i=1 \\ j \neq i}}^{N} \frac{\tau - \tau_i}{\tau_j - \tau_i}, \quad j = 1, \ldots, N,$$

and define the $N \times N$ matrix $\mathbf{B}$ by

$$b_{ij} = \int_{\tau_i}^{+1} L_j^\dagger(t)\; dt.$$

By the definition of $\mathbf{B}$, it follows that

$$\int_{\tau_i}^{+1} q(t)\, dt = (\mathbf{B}\mathbf{q})_i. \tag{42}$$

Combining Eq. (42) with (40) and (41), we obtain

$$\sum_{j=1}^{N}\sum_{i=1}^{N} w_j q_j A_{ji} p_i = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i p_i B_{ij} q_j.$$

Rearranging this last expression gives

$$\sum_{j=1}^{N}\sum_{i=1}^{N} q_j [w_j A_{ji} - w_i B_{ij}] p_i = 0.$$

Since this last result must hold for all $\mathbf{p}$ and $\mathbf{q}$, we conclude that the bracketed expression must vanish. Therefore,

$$B_{ij} = \frac{w_j}{w_i} A_{ji},$$

which shows that $\mathbf{B} = \mathbf{A}^\dagger$. Consequently, Eq. (42) yields (38).

Given $\mathbf{p} \in \mathbb{R}^N$, let $p(\tau)$ denote the polynomial of degree at most $N-1$ that satisfies $p(\tau_i) = p_i$. Let q be the polynomial of degree at most N defined by

$$q(\tau) = \int_{\tau}^{+1} p(t)\; dt. \tag{43}$$

Let $\mathbf{q} \in \mathbb{R}^{N+1}$ be the vector with components $q_i = q(\tau_i)$, $1 \le i \le N+1$. By Theorem 1 in Ref. 10 and by Eq. (43), we have

$$(\mathbf{D}_{1:N+1}^\dagger \mathbf{q})_i = \dot{q}(\tau_i) = -p(\tau_i) = -p_i, \quad 1 \le i \le N. \tag{44}$$

By (38), we have $\mathbf{q}_{1:N} = \mathbf{A}^\dagger \mathbf{p}$. Since $q_{N+1} = 0$, it follows that

$$\mathbf{D}_{1:N+1}^\dagger \mathbf{q} = \mathbf{D}_{1:N}^\dagger \mathbf{q}_{1:N} = \mathbf{D}_{1:N}^\dagger \mathbf{A}^\dagger \mathbf{p}. \tag{45}$$

Combining Eqs. (44) and (45) yields

$$\mathbf{D}_{1:N}^{\dagger}\mathbf{A}^{\dagger}\mathbf{p} = -\mathbf{p}.$$

Since $\mathbf{p}$ was arbitrary, we deduce that $\mathbf{D}_{1:N}^{\dagger}$ is invertible and $(\mathbf{D}_{1:N}^{\dagger})^{-1} = -\mathbf{A}^{\dagger}$. □

5 Costate Approximation Using Integral Legendre-Gauss-Radau Collocation

In this section we develop the relation between the multipliers arising in the integral Legendre-Gauss-Radau collocation scheme and the costate associated with the differential Legendre-Gauss-Radau collocation scheme. In Section 5.1 we review the differential form of the LGR collocation method, and in Section 5.2 we review the first-order optimality conditions for the discrete problem (see Refs. 9–11). In Section 5.3 we describe the integral form of the LGR collocation method, and we provide the first-order optimality conditions of the nonlinear programming problem described in Section 5.1, and the relationships between the multipliers in the integral and differential discretizations.

5.1 Differential Form of Legendre-Gauss-Radau Collocation

We will consider the so-called flipped LGR points located on the half-open interval $(-1, +1]$, as in Ref. 10, since the algebraic manipulations are somewhat simpler than those for the LGR points located on $[-1, +1)$ considered in Ref. 9. If $(\tau_1, \ldots, \tau_N)$ are the LGR quadrature points with $\tau_N = +1$, then the state is approximated as

$$\mathbf{y}(\tau) \approx \mathbf{Y}(\tau) = \sum_{i=0}^{N} \mathbf{Y}_i L_i(\tau), \quad L_i(\tau) = \prod_{\substack{j=0 \\ j\neq i}}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \tag{46}$$

where $\tau_0 = -1$ is the starting time and $L_i(\tau),\ i = 0, \ldots, N$, is a basis of Lagrange polynomials of degree N with support points $(\tau_0, \ldots, \tau_N)$. The time derivative of the state approximation at $\tau = \tau_i$, $1 \leq i \leq N$, is

$$\dot{\mathbf{y}}(\tau_i) \approx \dot{\mathbf{Y}}(\tau_i) = \sum_{j=0}^{N} \mathbf{Y}_j \dot{L}_j(\tau_i) = \left[\mathbf{D}\mathbf{Y}_{0:N}\right]_i, \tag{47}$$

where $\mathbf{Y}_i = \mathbf{Y}(\tau_i)$ and $\mathbf{D}$ is the $N \times (N+1)$ *Legendre-Gauss-Radau (LGR) differentiation matrix* whose elements are given by $D_{ij} = \dot{L}_j\left(\tau_i\right)$. Note that we collocate the dynamics at the quadrature points τ_i, $1 \leq i \leq N$, which includes the final time $\tau_1 = +1$, but not the initial time $\tau_0 = -1$. If $\mathbf{w} = (w_1, \ldots, w_N)$ is the vector of LGR quadrature weights, then the discretized control problem is

$$\min \quad J = \Phi(\mathbf{Y}_N) + \sum_{j=1}^{N} w_j g(\mathbf{Y}_j, \mathbf{U}_j), \tag{48}$$

subject to the collocated dynamics

$$\mathbf{D}\mathbf{Y}_{0:N} - \mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}) = \mathbf{0}, \tag{49}$$

$$\mathbf{Y}_0 = \mathbf{y}_0. \tag{50}$$

The NLP described by Eqs. (48)–(50) will be referred to as the *differential Legendre-Gauss-Radau (LGR) method*.

5.2 KKT Conditions Using Differential Legendre-Gauss-Radau Collocation

In Ref. 10 it is shown that the KKT first-order optimality conditions of the differential LGR collocation method can be written in the following form:

$$
\begin{aligned}
\mathbf{D}\mathbf{Y}_{0:N} &= \mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}), \quad \mathbf{Y}_0 = \mathbf{y}_0, & (51)\\
\mathbf{0} &= \nabla_u \mathcal{H}(\mathbf{Y}_i, \mathbf{U}_i, \boldsymbol{\lambda}_i), & (52)\\
(\mathbf{D}^\dagger \boldsymbol{\lambda}_{1:N})_i &= -\nabla_y \mathcal{H}(\mathbf{Y}_i, \mathbf{U}_i, \boldsymbol{\lambda}_i) + \frac{1}{w_N}\delta_{iN}(\boldsymbol{\lambda}_N - \nabla\Phi(\mathbf{Y}_N)), & (53)
\end{aligned}
$$

where $\boldsymbol{\lambda}_i$, $1 \leq i \leq N$, is the transformed multiplier associated with the collocated dynamics at τ_i, δ_{iN} is the Kronecker delta which is zero except for $\delta_{NN} = 1$, and $\mathbf{D}^\dagger$ is the $N \times N$ matrix defined by

$$
D^\dagger_{NN} = -D_{NN} + \frac{1}{w_N} \quad \text{and} \quad D^\dagger_{ij} = -\frac{w_j}{w_i} D_{ji} \quad \text{otherwise.} \tag{54}
$$

By Theorem 1 in Ref. 10, $\mathbf{D}^\dagger$ is a differentiation matrix for the space of polynomials of degree $N-1$. More precisely, if b is a polynomial of degree at most $N-1$ and $\mathbf{b} \in \mathbb{R}^N$ is the vector with i^{th} element $b_i = b(\tau_i)$ for $1 \leq i \leq N$, then

$$
(\mathbf{D}^\dagger \mathbf{b})_i = \dot{b}(\tau_i).
$$

It is noted that in Eqs. (51)–(53), the time derivative of the state is approximated using the differentiation matrix $\mathbf{D}$ for the space of polynomials of degree N [see Eq. (47)], while the costate is being differentiated by a differentiation matrix $\mathbf{D}^\dagger$ for the space of polynomials of degree $N-1$.

5.3 Integral Form of Legendre-Gauss-Radau Collocation

It is shown in Ref. 10 that the LGR differentiation matrix $\mathbf{D}$ given by Eq. (47) has the property that the square matrix $\mathbf{D}_{1:N}$ obtained by removing the first column of $\mathbf{D}$ is full-rank and $-\mathbf{D}_{1:N}^{-1}\mathbf{D}_0 = \mathbf{1}$. Using these properties, the dynamic constraints, Eq. (5) can be approximated as

$$
\mathbf{Y}_{1:N} = \mathbf{1}\mathbf{y}_0 + \mathbf{A}\mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}), \tag{55}
$$

where $\mathbf{1}$ as a column vector of all ones. The NLP described by Eqs. (48) and (55) will be referred to as the *integral Legendre-Gauss-Radau (LGR) collocation method*.

The KKT first-order optimality conditions of the integral LGR collocation method are found by taking the partial derivatives of the Lagrangian of the NLP with respect to every free variable and setting the result to zero. The Lagrangian is

$$
\mathcal{L} = \Phi(\mathbf{Y}_N) + \sum_{i=1}^{N} w_i g(\mathbf{Y}_i, \mathbf{U}_i) + \langle \mathbf{R}, \mathbf{1}\mathbf{y}_0 + \mathbf{A}\mathbf{f}(\mathbf{Y}_{1:N}, \mathbf{U}_{1:N}) - \mathbf{Y}_{1:N} \rangle,
$$

where $\mathbf{R}_i$, $1 \leq i \leq N$, is the multiplier associated with the collocated dynamics at τ_i. The partial derivatives with respect to the control and the state yield the relations

$$\begin{aligned} \mathbf{0} &= w_i \nabla_u g(\mathbf{Y}_i, \mathbf{U}_i) + \nabla_u \langle (\mathbf{A}^\mathsf{T}\mathbf{R})_i, \mathbf{f}(\mathbf{Y}_i, \mathbf{U}_i) \rangle, & (56) \\ \mathbf{R}_i &= w_i \nabla_y g(\mathbf{Y}_i, \mathbf{U}_i) + \nabla_y \langle (\mathbf{A}^\mathsf{T}\mathbf{R})_i, \mathbf{f}(\mathbf{Y}_i, \mathbf{U}_i) \rangle + \delta_{Ni} \nabla \Phi(\mathbf{Y}_N), & (57) \end{aligned}$$

$1 \leq i \leq N$. Next, we make the change of variables

$$\mathbf{r}_i = \mathbf{R}_i / w_i - (\delta_{Ni}/w_i) \nabla \Phi(\mathbf{Y}_N). \tag{58}$$

In addition, we define the matrix $\mathbf{A}^\dagger$ as

$$A^\dagger_{ij} = \frac{w_j}{w_i} A_{ji}. \tag{59}$$

Substituting the results of Eqs. (58) and (59) into (56) and (57) and dividing the i^{th} equation by w_i, we obtain

$$\begin{aligned} \mathbf{0} &= \nabla_u \mathcal{H}(\mathbf{Y}_i, \mathbf{U}_i, (\mathbf{A}^\dagger \mathbf{r})_i + (A_{Ni}/w_i) \nabla \Phi(\mathbf{Y}_N)), \\ \mathbf{r}_i &= \nabla_y \mathcal{H}(\mathbf{Y}_i, \mathbf{U}_i, (\mathbf{A}^\dagger \mathbf{r})_i + (A_{Ni}/w_i) \nabla \Phi(\mathbf{Y}_N)). \end{aligned}$$

In Ref. 10, it is shown that

$$A_{ij} = \int_{-1}^{\tau_i} L^\dagger_j(\tau), \qquad L^\dagger_j = \prod_{\substack{i=1 \\ j \neq i}}^{N} \frac{\tau - \tau_i}{\tau_j - \tau_i}.$$

Because $\tau_N = +1$, we deduce that $A_{Ni} = w_i$ and $A_{Ni}/w_i = 1$. Hence, we obtain the following necessary optimality conditions

$$\begin{aligned} \mathbf{0} &= \nabla_u \mathcal{H}(\mathbf{Y}_i, \mathbf{U}_i, \boldsymbol{\mu}_i), & (60) \\ \mathbf{r}_i &= \nabla_y \mathcal{H}(\mathbf{Y}_i, \mathbf{U}_i, \boldsymbol{\mu}_i), & (61) \end{aligned}$$

$1 \leq i \leq N$, where

$$\boldsymbol{\mu}_i = \nabla \Phi(\mathbf{Y}_N) + (\mathbf{A}^\dagger \mathbf{r})_i. \tag{62}$$

Comparing Eqs. (52) and (53) to Eqs. (60) and (61), we see that they are equivalent if the following conditions hold:

$$\text{(a)} \;\; \boldsymbol{\lambda} = \mathbf{1} \nabla \Phi(\mathbf{Y}_N) + \mathbf{A}^\dagger \mathbf{r} \quad \text{and} \quad \text{(b)} \;\; \mathbf{r} = \left(\frac{1}{w_N} \mathbf{e}_N \mathbf{e}_N^\mathsf{T} - \mathbf{D}^\dagger \right) \boldsymbol{\lambda} - \frac{1}{w_N} \mathbf{e}_N \nabla \Phi(\mathbf{Y}_N),$$

where $\mathbf{e}_N$ is the last column of the identity (the column vector whose entries are all zero except the last entry which is 1). The conditions (a) and (b) are equivalent in that if (a) holds, then so does (b) and if (b) holds, then so does (a). This equivalence is based on the following analogue of Theorem 1 which was established in Ref. 34.

Theorem 2. *Let M_j denote the Lagrange interpolating polynomials defined by*

$$M_j(\tau) = \prod_{\substack{i=1 \\ i \neq j}}^{N-1} \frac{\tau - \tau_i}{\tau_j - \tau_i}, \quad 1 \le j \le N-1.$$

The first $N-1$ columns of $\mathbf{A}^\dagger$ are given by

$$A^\dagger_{ij} = \int_{\tau_i}^{+1} M_j(\tau)\, d\tau - w_N M_j(\tau_N), \quad 1 \le i \le N,$$

while all entries in the last column of $\mathbf{A}^\dagger$ are w_N. Moreover, we have

$$\mathbf{A}^\dagger = \left(\frac{1}{w_N} \mathbf{e}_N \mathbf{e}_N^\mathsf{T} - \mathbf{D}^\dagger \right)^{-1}.$$

Based on Theorem 2 and the special form of the last column of $\mathbf{A}^\dagger$, if (b) holds, then we can multiply both sides of (b) by $\mathbf{A}^\dagger$ to obtain

$$\mathbf{A}^\dagger \mathbf{r} = \boldsymbol{\lambda} - \mathbf{1}\nabla\Phi(\mathbf{Y}_N),$$

which yields (a). Another consequence of the theorem is that

$$\mathbf{A}^\dagger \mathbf{r} = w_N(r_N - q(+1)) + \int_{\tau_i}^{+1} q(\tau)\, d\tau,$$

where q is the polynomial of degree at most $N-2$ defined by $q(\tau_i) = r_i$ for $1 \le i \le N-1$. Hence, $\boldsymbol{\mu}_i$ in Eqs. (62) represents an approximation to the continuous costate $\boldsymbol{\lambda}(\tau_i)$ of Eq. (17) in which the integral is replaced by a quadrature based on function values at τ_1, ..., τ_{N-1} plus an additional term connected with the difference between $\mathbf{r}_N$ and the polynomial extrapolation of $(\mathbf{r}_1, \ldots, \mathbf{r}_{N-1})$ to $\tau = +1$. Clearly, the Radau approximation to the control problem with integrated dynamics is nontrivial.

6 Examples

We now consider two examples that employ both the Legendre-Gauss and Legendre-Gauss-Radau methods developed in Sections 4 and 5. The first example is an initial-value optimal control problem with a Mayer cost while the second example is a boundary-value optimal control problem with a Lagrange cost. The examples provides an analysis of the errors in both the integral costate approximation, $\mathbf{r}$, and the differential costate estimate approximation, $\boldsymbol{\lambda}$.

7 Example 1

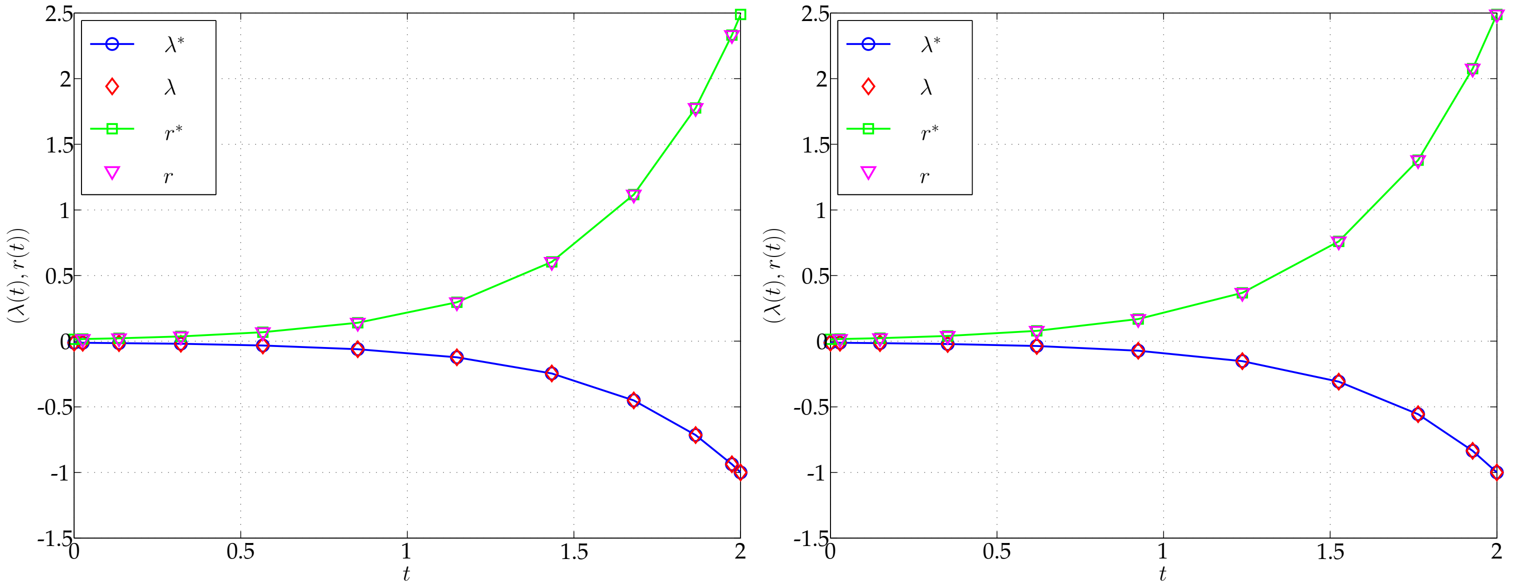
Consider the following optimal control problem:

$$\text{Minimize } J = -y(2) \text{ subject to } \begin{cases} \dot{y}(t) &= \frac{5}{2}(-y(t) + y(t)u(t) - u(t)^2), \\ y(0) &= 1. \end{cases} \tag{63}$$

The solution to the optimal control problem given in Eq. (63) is

$$\begin{aligned} y^*(t) &= \frac{4}{a(t)} &, \quad u^*(t) &= \frac{y^*(t)}{2}, \\ r^*(t) &= -\frac{(15\exp(5t/2) - 1)(-3\exp(5t/2) - 1)}{(2b\exp(5t/2))} &, \quad \lambda^*(t) &= -\frac{\exp(2\ln(a(t)) - 5t/2)}{b}, \end{aligned}$$

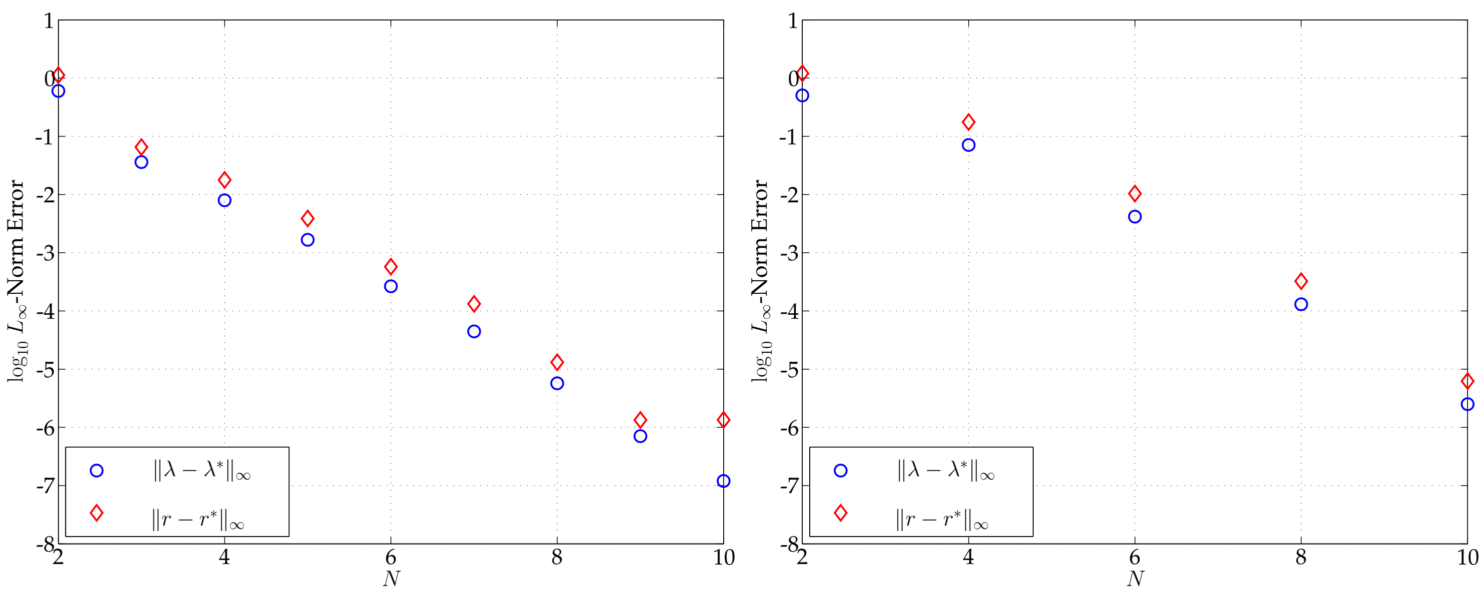
where $a(t) = 1+3\exp(5t/2)$, and $b = \exp(-5)+6+9\exp(5)$. The example was solved using the integral LG and LGR collocation methods and the NLP solver SNOPT,[1] where SNOPT was implemented using optimality and feasibility tolerances of 1×10^{-7} and 2×10^{-7}, respectively. Figures 1a and Figures 1b show the LG and LGR approximations, respectively, of the dual variables λ and r alongside the optimal values p^* and λ^* for $N = 10$ collocation points, where it is seen that the approximations are indistinguishable from the optimal values. Next, Figs. 2a and Figures 2b shows the base ten logarithm of the L_∞-norm errors in λ and p as a function of the number of collocation points, N for $N = (2, 4, \ldots, 10)$. It is seen for this example that the errors in both λ and r decrease exponentially and in all cases the approximation of λ is slightly more accurate than the approximation of r. Finally, it is observed that, for any particular value of N, the LG costate approximation is one order of magnitude more accurate than the LGR costate estimate.



(a) LG Differential and Integral Costate Approximation for $N = 10$.

(b) LGR Differential and Integral Costate Approximations for $N = 10$.

Figure 1: Exact and Approximated Differential and Integral Costate for Example 1 for $N = 10$ LG and LGR Collocation Points.



(a) LG Base Ten Logarithm of L_∞-Norm Error.

(b) LGR Base Ten Logarithm of L_∞-Norm Error.

Figure 2: Base Ten Logarithm of L_∞-Norm Error of Differential and Integral Costate for Example 1.

7.1 Example 2

Consider the following optimal control problem:

$$\text{Minimize } J = \int_0^{tf} (\log^2 y + u^2)dt \text{ subject to } \begin{cases} \dot{y}(t) & = & y(t)\log y(t) + y(t)u(t), \\ y(0) & = & 5, \\ y(t_f) & = & 3, \\ t_f & = & 5. \end{cases} \tag{64}$$

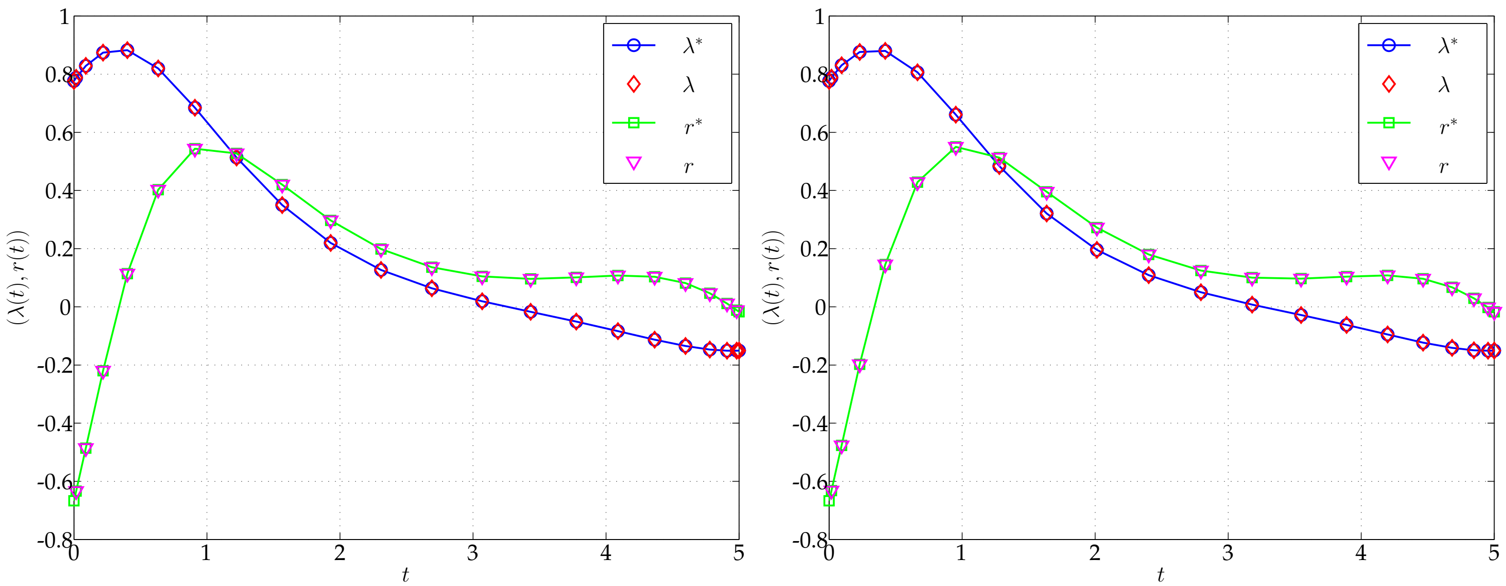
The solution to the optimal control problem in Eq. (64) is

$$\begin{array}{rclcrcl} y^*(t) & = & \exp(x^*(t)) & , & u^*(t) & = & -\psi_*(t), \\ r^*(t) & = & \exp(-x^*(t))(\dot{x}^*(t)\psi^*(t) - \dot{\psi}^*(t)) & , & \lambda^*(t) & = & \exp(-x^*(t))\psi^*(t), \end{array}$$

where

$$\begin{bmatrix} x^*(t) \\ \psi^*(t) \end{bmatrix} = \begin{bmatrix} \exp(-t\sqrt{2}) & \exp(t\sqrt{2}) \\ (1+\sqrt{2})\exp(-t\sqrt{2}) & (1-\sqrt{2})\exp(t\sqrt{2}) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \exp(-t_f\sqrt{2}) & \exp(t_f\sqrt{2}) \end{bmatrix} \begin{bmatrix} \log y_0 \\ \log y_f \end{bmatrix}.$$

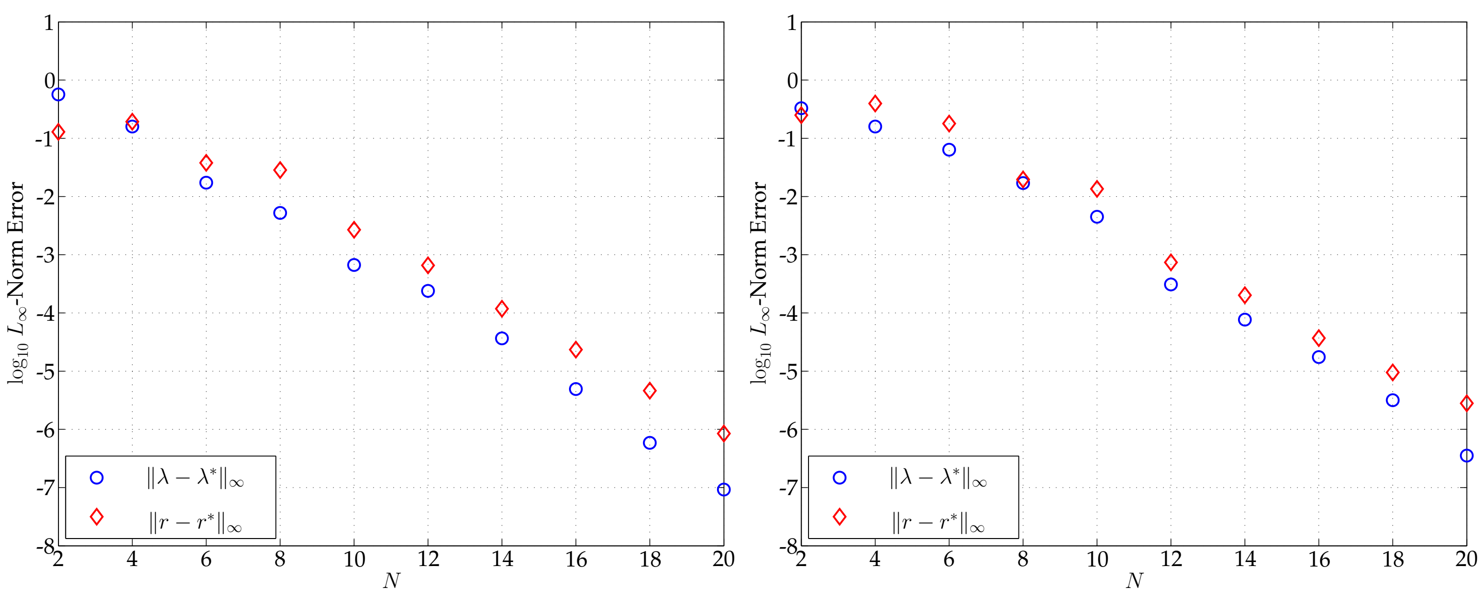
The example was solved by the integral LG and LGR collocation methods using the NLP solver SNOPT[1] using optimality and feasibility tolerances of 1×10^{-7} and 2×10^{-7}, respectively. Figures 1a and Figures 1b show the LG and LGR approximations, respectively, of the dual variables λ and r alongside the optimal values p^* and λ^* for $N = 20$ collocation points, where it is seen that the approximations are indistinguishable from the optimal values. Next, Figs. 4a and Figures 4b shows the base ten logarithm of the L_∞-norm errors in λ and r as a function of the number of collocation points, N for $N = (2, 4, \dots, 20)$. Similar to the results obtained in the first example, it is seen for this example that the errors in both λ and r decrease exponentially and in all cases the approximation of λ is slightly better than the approximation of r. In addition, and again similar to the results obtained in the first example, it is observed for any particular value of N that the LG costate approximation is one order of magnitude more accurate than the LGR costate approximation. Note that since this example includes a terminal constraint, we need to introduce a Lagrange multiplier for the constraint which is added to the terminal multipliers for both the continuous and discrete problems.



(a) LG Differential and Integral Costate Approximations for $N = 32$.

(b) LGR Differential and Integral Costate Approximations for $N = 20$.

Figure 3: Differential and Integral Costate Approximations For Example 2 for $N = 20$ LG and LGR Collocation Points.



(a) Base Ten Logarithm of LG L_∞-Norm Error.

(b) Base Ten Logarithm of LGR L_∞-Norm Error.

Figure 4: Base Ten Logarithm of L_∞-Norm Error of Differential and Integral Costate for Example 2.

8 Conclusions

Two methods have been presented for approximating the costate of an optimal control problem using the integral form of orthogonal collocation at Legendre-Gauss and Legendre-Gauss-Radau points. A new dual variable called the integral costate has been introduced in order to obtain the first-order optimality conditions when the continuous-time Bolza optimal control problem is written in integral form. It was shown that discrete forms of the integral costate are related to the costate of the original differential form of the problem via integration matrices. The integral Legendre-Gauss collocation method produces a costate that is approximated by a polynomial of degree N while the integral Legendre-Gauss-Radau method produces a costate that is approximated by a polynomial of degree $N-1$. The relationship between the costate of the original optimal control problem and the integral costate associated with the integral form of the optimal control problem provides an equivalence between the first-order optimality conditions of the differential and integral forms of the optimal control problem. Finally, it is shown on two examples that the Legendre-Gauss and Legendre-Gauss-Radau costate approximations converge exponentially.

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Appendix

The first-order optimality conditions of the integral form of the optimal control problem defined in Eqs. (4)–(6) are now derived. It is assumed that all the functions Φ, g, and f are continuously differentiable, and we assume that $(\mathbf{y},\mathbf{u})$ is a smooth local minimizer. Let ϵ denote a scalar, and consider the perturbed control $\mathbf{u}+\epsilon\mathbf{v}$ where $\mathbf{v}$ is an arbitrary smooth function. For ϵ near zero, the corresponding state is the solution of

$$\mathbf{y}_\epsilon(\tau) = \mathbf{y}_0 + \int_{-1}^{\tau} \mathbf{f}(\mathbf{y}_\epsilon(\tau)), \mathbf{u}(\tau) + \epsilon\mathbf{v}(\tau))\ dt, \quad \tau \in [-1,+1]. \tag{65}$$

The objective function associated with the perturbation $\epsilon\mathbf{v}$ is

$$J_\epsilon = \Phi(\mathbf{y}_\epsilon(+1)) + \int_{-1}^{+1} g(\mathbf{y}_\epsilon(\tau), \mathbf{u}(\tau) + \epsilon\mathbf{v}(\tau))d\tau.$$

Since $(\mathbf{y},\mathbf{u})$ is a local minimizer, the objective function value for $\epsilon \neq 0$ can only be larger than the objection function value for $\epsilon = 0$. In other words, as a function of ϵ, the objective function achieves a local minimum at $\epsilon = 0$. The first-order necessary optimality condition is that the derivative of J_ϵ should vanish at $\epsilon = 0$. By the chain rule, we have

$$\left.\frac{dJ_\epsilon}{d\epsilon}\right|_{\epsilon=0} = \nabla\Phi(y(+1))\ \left.\frac{d\mathbf{y}_\epsilon}{d\epsilon}(+1)\right|_{\epsilon=0} + \int_{-1}^{+1} g_y(\tau)\ \left.\frac{d\mathbf{y}_\epsilon}{d\epsilon}(\tau)\right|_{\epsilon=0} + g_u(\tau)\mathbf{v}(\tau)\ d\tau = 0, \tag{66}$$

where $g_y(\tau) = \nabla_y g(\mathbf{y}(\tau), \mathbf{u}(\tau))$ and $g_u(\tau) = \nabla_u g(\mathbf{y}(\tau), \mathbf{u}(\tau))$. Here, the vectors $\mathbf{y}$ and $\mathbf{v}$ are column vectors. We then obtain a formula for the derivative of $\mathbf{y}_\epsilon$ by differentiating Eq. (65) with respective ϵ. Let $\mathbf{z}$ denote this derivative:

$$\mathbf{z}(\tau) = \left.\frac{d\mathbf{y}_\epsilon}{d\epsilon}(\tau)\right|_{\epsilon=0} \tag{67}$$

Differentiating Eq. (65) using the chain rule yields

$$\mathbf{z}(\tau) = \int_{-1}^{\tau} \mathbf{f}_y(t)\mathbf{z}(t) + \mathbf{f}_u(t)\mathbf{v}(t)\ dt, \quad \tau \in [-1,+1], \tag{68}$$

where $\mathbf{f}_y(t) = \nabla_y \mathbf{f}(\mathbf{y}(t), \mathbf{u}(t))$ and $\mathbf{f}_u(t) = \nabla_u \mathbf{f}(\mathbf{y}(t), \mathbf{u}(t))$. Here, $\mathbf{z}$ is also a column vector. After replacing the derivative of $\mathbf{y}_\epsilon$ by $\mathbf{z}$ in Eq. (66), we obtain

$$\nabla\Phi(\mathbf{y}(+1))\mathbf{z}(+1) + \int_{-1}^{+1} g_y(\tau)\mathbf{z}(\tau) + g_u(\tau)\mathbf{v}(\tau)\, d\tau = 0. \tag{69}$$

By Eq. (68), it follows that

$$\mathbf{z}(+1) = \int_{-1}^{+1} \mathbf{f}_y(\tau)\mathbf{z}(\tau) + \mathbf{f}_u(\tau)\mathbf{v}(\tau)\, d\tau. \tag{70}$$

Let $\mathbf{p}$ be any smooth row vector. Multiplying Eq. (68) by $\mathbf{p}$ and integrating over τ between -1 and $+1$, we obtain

$$\int_{-1}^{+1} \mathbf{p}(\tau) \int_{-1}^{\tau} \mathbf{f}_y(t)\mathbf{z}(t) + \mathbf{f}_u(t)\mathbf{v}(t)\, dt d\tau - \int_{-1}^{+1} \mathbf{p}(\tau)\mathbf{z}(\tau)\, d\tau = 0.$$

In the double integral, we change the order of integration and then interchange the dummy variables t and τ to get

$$\int_{-1}^{+1} \left[\int_{\tau}^{+1} \mathbf{p}(t)\, dt\right] (\mathbf{f}_y(\tau)\mathbf{z}(\tau) + \mathbf{f}_u(\tau)\mathbf{v}(\tau))\, d\tau - \int_{-1}^{+1} \mathbf{p}(\tau)\mathbf{z}(\tau)\, d\tau = 0. \tag{71}$$

Adding Eq. (71) to Eq. (69) and substituting for $\mathbf{z}(+1)$ using Eq. (70). we obtain

$$\begin{aligned} 0 \;=\; & \int_{-1}^{+1} \left(g_y(\tau) - \mathbf{p}(\tau) + \left[\nabla\Phi(\mathbf{y}(+1)) + \int_{\tau}^{+1} \mathbf{p}(t)\, dt\right] \mathbf{f}_y(\tau) \right) \mathbf{z}(\tau)\, d\tau \\ & + \int_{\tau}^{+1} \left(g_u(\tau) + \left[\nabla\Phi(\mathbf{y}(+1)) + \int_{\tau}^{+1} \mathbf{p}(t)\, dt\right] \mathbf{f}_u(\tau) \right) \mathbf{v}(\tau)\, d\tau. \end{aligned} \tag{72}$$

We now choose $\mathbf{p}$ so that the coefficient of $\mathbf{z}$ vanishes. In other words,

$$\mathbf{p}(\tau) = g_y(\tau) + \left[\nabla\Phi(\mathbf{y}(+1)) + \int_{\tau}^{+1} \mathbf{p}(t)\, dt\right] \mathbf{f}_y(\tau). \tag{73}$$

With this choice for $\mathbf{p}$, Eq. (72) yields

$$\int_{\tau}^{+1} \left(g_u(\tau) + \left[\nabla\Phi(\mathbf{y}(+1)) + \int_{\tau}^{+1} \mathbf{p}(t)\, dt\right] \mathbf{f}_u(\tau) \right) \mathbf{v}(\tau)\, d\tau = 0.$$

Since $\mathbf{v}$ was arbitrary, the coefficient of $\mathbf{v}$ vanishes, and we have

$$g_u(\tau) + \left[\nabla\Phi(\mathbf{y}(+1)) + \int_{\tau}^{+1} \mathbf{p}(t)\, dt\right] \mathbf{f}_u(\tau) = 0 \tag{74}$$

for all $\tau \in [-1, +1]$. In terms of the Hamiltonian $\mathcal{H}(\mathbf{y}, \mathbf{u}, \boldsymbol{\lambda}) = g(\mathbf{y}, \mathbf{u}) + \langle \boldsymbol{\lambda}, \mathbf{f}(\mathbf{y}, \mathbf{u}) \rangle$, Eq. (73) can be expressed

$$\mathbf{p}(\tau) = \nabla_y \mathcal{H}(\mathbf{y}(\tau), \mathbf{u}(\tau), \boldsymbol{\lambda}(\tau)), \quad \boldsymbol{\lambda}(\tau) = \nabla\Phi(\mathbf{y}(+1)) + \int_{\tau}^{+1} \mathbf{p}(t)\, dt. \tag{75}$$

Similarly, Eq. (74) is equivalent to

$$\nabla_u \mathcal{H}(\mathbf{y}(\tau), \mathbf{u}(\tau), \boldsymbol{\lambda}(\tau)) = \mathbf{0}, \quad \boldsymbol{\lambda}(\tau) = \nabla\Phi(\mathbf{y}(+1)) + \int_{\tau}^{+1} \mathbf{p}(t)\, dt. \tag{76}$$

Our analysis has established the existence of a function $\mathbf{p}$ which satisfies the first-order optimality conditions of Eqs. (75) and (76). We now show the relationship between $\mathbf{p}$ and a Lagrange multiplier. Using the integral dynamics of Eq. (5), the objective function of our control problem can be expressed

$$\Phi\left(\mathbf{y}_0 + \int_{-1}^{+1} \mathbf{f}(\mathbf{y}(\tau), \mathbf{u}(\tau))\, dt\right) + \int_{-1}^{+1} g(\mathbf{y}(\tau), \mathbf{u}(\tau)) d\tau. \tag{77}$$

We multiply the integral dynamics by $\mathbf{p}(\tau)$, integrate over $\tau \in [-1, +1]$, and add to the objective function to obtain the Lagrangian

$$\begin{aligned}\mathcal{L}(\mathbf{y}, \mathbf{u}, \mathbf{p}) &= \Phi\left(\mathbf{y}_0 + \int_{-1}^{+1} \mathbf{f}(\mathbf{y}(\tau), \mathbf{u}(\tau))\, dt\right) + \int_{-1}^{+1} g(\mathbf{y}(\tau), \mathbf{u}(\tau)) + \mathbf{p}(\tau)(\mathbf{y}_0 - \mathbf{y}(\tau))\, d\tau \\ &\quad + \int_{-1}^{+1} \int_{-1}^{\tau} \mathbf{p}(\tau)\mathbf{f}(\mathbf{y}(t), \mathbf{u}(t))\, dt d\tau.\end{aligned}$$

Similar to Eq. (71), we change the order of integration in last term and we interchange the dummy variables t and τ to rewrite this as

$$\begin{aligned}\mathcal{L}(\mathbf{y}, \mathbf{u}, \mathbf{p}) &= \Phi\left(\mathbf{y}_0 + \int_{-1}^{+1} \mathbf{f}(\mathbf{y}(\tau), \mathbf{u}(\tau))\, dt\right) \\ &\quad + \int_{-1}^{+1} \left(g(\mathbf{y}(\tau), \mathbf{u}(\tau)) + \mathbf{p}(\tau)(\mathbf{y}_0 - \mathbf{y}(\tau)) + \left[\int_{\tau}^{+1} \mathbf{p}(t)\, dt\right] \mathbf{f}(\mathbf{y}(\tau), \mathbf{u}(\tau))\right)\, d\tau.\end{aligned}$$

Equating to zero the Fréchet derivatives of $\mathcal{L}$ with respect to $\mathbf{y}$ and $\mathbf{u}$, we obtain, respectively, Eq. (73) [or equivalently Eq. (75)] and Eq. (74) [or equivalently Eq. (76)]. Therefore, $\mathbf{p}$ can be viewed as a Lagrange multiplier for the integral dynamics when the objective function is written as given in Eq. (77).