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Direct Trajectory Optimization and Costate Estimation of State Inequality Path-Constrained Optimal Control Problems Using a Radau Pseudospectral Method

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A Radau pseudospectral method is derived for solving state-inequality path constrained optimal control problems. The continuous-time state-inequality path constrained optimal control problem is modified by applying a set of tangency conditions at the entrance of the activity of the path constraint. It is shown that the first-order optimality condition of the nonlinear programming problem associated with the Radau pseudospectral method is equivalent to the Radau pseudospectral discretized first-order optimality conditions of the modified continuous-time optimal control problem. The method is applied to a classical state-inequality path constrained optimal control problem and it is found that the solution accuracy is improved significantly when compared with the accuracy of the solution obtained using the original Radau pseudospectral discretization.

I. Introduction

Over the past two decades direct collocation methods have become increasingly popular in the numerical solution of complex constrained optimal control problems because they avoid many of the limitations associated with indirect methods. In even more recent years, a great deal of research has been done on the class of direct collocation *pseudospectral methods*.[3,12,16,18–21] In a pseudospectral method, the state is approximated using a basis of either Lagrange of Chebyshev polynomials and the dynamics are collocated at points associated with a Gaussian quadrature. The most common collocation points, which are the roots of a linear combination of Legendre polynomials or derivatives of Legendre polynomials, are *Legendre-Gauss* (LG), *Legendre-Gauss-Radau* (LGR), and *Legendre-Gauss-Lobatto* (LGL) points. All three types of Legendre-Gauss quadrature points are defined on the domain $\tau \in [-1, 1]$, but differ significantly in that the LG points include *neither* of the endpoints, the LGR points include *one* of the endpoints, and the LGL points include *both* of the endpoints. In addition, the LGR points are asymmetric relative to the origin and are not unique in that they can be defined using either the initial point or the terminal point.

One important class of optimal control problems that can pose significant computational challenges using either an indirect or direct method are problems with active inequality path constraints.[26,30] Problems with active state- and control-inequality path constraints can result in a discontinuous optimal control, while problems with state-inequality path constraints can result in

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a non-smooth state and/or a discontinuous costate. When an inequality path constrained optimal control problem is solved using an indirect method, the first-order optimality conditions from the calculus of variations must be modified in order to properly account for possible discontinuities in the optimal solution. On the other hand, when a direct method is used to solve an inequality path-constrained optimal control problem, the state is generally approximated using a piecewise smooth function and any possible discontinuities may not be located at a mesh point that defines the junction between two piecewise smooth segments.

In this paper we develop a new direct collocation approach for accurately solving continuous-time optimal control problems with state-inequality path constraints. The approach is motivated by the results of Ref. 35 where it was shown that the accuracy of the pseudospectral costate estimate can be quite poor due to discontinuities that arise in the presence of active state-inequality path constraints. The method developed in this paper utilizes a modified version of the Radau pseudospectral method.[18–20] Specifically, the optimal control problem is divided into a mesh such that the times of the mesh points are included as variables in the optimal control problem. This leads to an optimal control problem where it is desired to determine not only the optimal state and control in each mesh interval, but it is also desired to determine the optimal values of mesh point times. Furthermore, the standard Radau pseudospectral method is reformulated by including a set of tangency conditions[26]) that define the conditions at the start of the constrained arc.

II. Continuous Bolza Optimal Control Problem

Consider the following continuous-time state-inequality path constrained optimal control problem on the domain $t \in [t_0, t_f] = \mathcal{I}$, where $\mathcal{I}$ has been divided into K mesh intervals $\mathcal{S}_k = [T_{k-1}, T_k] \subseteq [t_0, t_f]$, $(k = 1, \ldots, K)$, where $T_0 = t_0$, $T_K = t_f$, $T_{k-1} < T_k$, $(k = 1, \ldots, K)$, and $\bigcup_{k=1}^{K} \mathcal{S}_k = \mathcal{I}$. Furthermore, without loss of generality we can transform the independent variable in each mesh interval from $t \in [T_{k-1}, T_k]$ to $\tau^{(k)} \in [-1, +1]$ via the affine transformation

$$t = \frac{T_k - T_{k-1}}{2}\tau^{(k)} + \frac{T_k + T_{k-1}}{2} \tag{1}$$

The optimal control problem is then stated as follows. Determine the state, $\mathbf{y}^{(k)}(\tau^{(k)}; T_{k-1}, T_k) \in \mathbb{R}^n$ and the control, $\mathbf{u}^{(k)}(\tau^{(k)}; T_{k-1}, T_k) \in \mathbb{R}^m$, in mesh intervals $k \in [1, \ldots, K]$ that minimize the cost functional

$$\begin{aligned} J &= \Phi(\mathbf{y}^{(1)}(T_0), T_0, \mathbf{y}^{(K)}(T_K), T_K) \\ &+ \sum_{k=1}^{K} \frac{T_k - T_{k-1}}{2} \int_{-1}^{+1} g(\mathbf{y}^{(k)}(\tau^{(k)}; T_{k-1}, T_k), \mathbf{u}^{(k)}(\tau^{(k)}; T_{k-1}, T_k), \tau^{(k)}; T_{k-1}, T_k) d\tau \end{aligned} \tag{2}$$

subject to the dynamic constraint

$$\begin{aligned} &\frac{T_k - T_{k-1}}{2}\mathbf{f}(\mathbf{y}^{(k)}(\tau^{(k)}; T_{k-1}, T_k), \mathbf{u}^{(k)}(\tau^{(k)}; T_{k-1}, T_k), \tau^{(k)}; T_{k-1}, T_k) \\ &- \nabla_\tau \mathbf{y}^{(k)}(\tau^{(k)}; T_{k-1}, T_k) = \mathbf{0}, \quad (k = 1, \ldots, K), \end{aligned} \tag{3}$$

the boundary condition

$$\boldsymbol{\phi}(\mathbf{y}^{(1)}(T_0), T_0, \mathbf{y}^{(K)}(T_K), T_K) = \mathbf{0}, \tag{4}$$

and the state-inequality path constraint

$$C(\mathbf{y}^{(k)}(\tau^{(k)}; T_{k-1}, T_k), \tau^{(k)}; T_{k-1}, T_k) \leq \mathbf{0}, \quad (k = 1, \ldots, K). \tag{5}$$

From this point forth the optimal control problem defined in Eqs. (2)–(5) will be denoted problem $\mathcal{P}$. We note that generality is not lost in problem $\mathcal{P}$ by considering a scalar state-inequality path constraint because our approach can be applied to a vector inequality path constraint by considering each component individually. Furthermore, we consider the specific case of three mesh intervals (that is, $K = 3$) such that on the optimal solution the state-inequality path constraint is inactive in the first mesh interval, active in the second interval, and again inactive in the third mesh interval, and the switch times in the path constraint occur at *unknown* times $T_1 \in I$ and $T_2 \in I$, $T_1 < T_2$. The state-inequality path constraint in problem $\mathcal{P}$ can be replaced by the conditions

$$\boldsymbol{\psi}(\mathbf{y}^{(2)}(T_1), T_1) = \mathbf{0}. \tag{6}$$

where

$$\boldsymbol{\psi}(\mathbf{y}^{(2)}(\tau^{(2)}; T_1, T_2), \tau; T_1, T_2) \equiv \begin{bmatrix} C(\mathbf{y}^{(2)}(\tau^{(2)}; T_1, T_2), \tau^{(2)}; T_1, T_2) \\ \nabla_\tau C(\mathbf{y}^{(2)}(\tau^{(2)}; T_1, T_2), \tau^{(2)}; T_1, T_2) \\ \vdots \\ \nabla_\tau^{q-1} C(\mathbf{y}^{(2)}(\tau^{(2)}; T_1, T_2), \tau^{(2)}; T_1, T_2) \end{bmatrix} \tag{7}$$

along with the following state and control equality path constraint:

$$\nabla_\tau^q C(\mathbf{y}^{(2)}(\tau^{(2)}; T_1, T_2), \mathbf{u}^{(2)}(\tau^{(2)}; T_1, T_2), \tau^{(2)}; T_1, T_2) = 0. \tag{8}$$

The method described in this paper is then developed using the following modification of problem $\mathcal{P}$. Minimize the cost functional of Eq. (2) subject to the dynamic constraint of Eq. (3), the boundary conditions of Eq. (4) and (6), and the path constraint of Eq. (8). This modified optimal control problem will be referred to henceforth as problem $\mathcal{M}$. It is seen that, unlike problem $\mathcal{P}$, problem $\mathcal{M}$ contains an interior-point constraint due to the tangency conditions that determine the start of the segment where the inequality path constraint is binding at the optimal solution.

The first-order optimality conditions of problem $\mathcal{M}$ obtained using the calculus of variations are given as[26]

$$\begin{aligned}
\mathbf{0} &= \nabla_u H^{(k)}, \quad (k = 1, 2, 3), && (9)\\
\nabla_\tau \boldsymbol{\lambda}^{(k)} &= -\frac{T_k - T_{k-1}}{2} \nabla_y H^{(k)}, \quad (k = 1, 2, 3), && (10)\\
\boldsymbol{\lambda}^{(1)}(T_0) &= -\nabla_{y^{(1)}(T_0)}(\boldsymbol{\Phi} - \langle \boldsymbol{\upsilon}, \boldsymbol{\phi} \rangle), && (11)\\
\boldsymbol{\lambda}^{(2)}(T_1) &= \boldsymbol{\lambda}^{(1)}(T_1) + \nabla_{y^{(1)}(T_1)} \langle \boldsymbol{\omega}, \boldsymbol{\psi} \rangle, && (12)\\
\boldsymbol{\lambda}^{(3)}(T_2) &= \boldsymbol{\lambda}^{(2)}(T_2), && (13)\\
\boldsymbol{\lambda}^{(3)}(T_3) &= \nabla_{y^{(3)}(T_3)}(\boldsymbol{\Phi} - \langle \boldsymbol{\upsilon}, \boldsymbol{\phi} \rangle), && (14)\\
H^{(1)}(T_0) &= \nabla_{T_0}(\boldsymbol{\Phi} - \langle \boldsymbol{\upsilon}, \boldsymbol{\phi} \rangle) = -\frac{T_1 - T_0}{2} \int_{-1}^{+1} \nabla_{T_0} H^{(1)} d\tau + \frac{1}{2} \int_{-1}^{+1} H^{(1)} d\tau, && (15)\\
H^{(2)}(T_1) &= H^{(1)}(T_1) - \nabla_{T_1} \langle \boldsymbol{\omega}, \boldsymbol{\psi} \rangle, && (16)\\
H^{(3)}(T_2) &= H^{(2)}(T_2), && (17)\\
H^{(3)}(T_3) &= -\nabla_{T_3}(\boldsymbol{\Phi} - \langle \boldsymbol{\upsilon}, \boldsymbol{\phi} \rangle) = \frac{T_3 - T_2}{2} \int_{-1}^{+1} \nabla_{T_3} H^{(3)} d\tau + \frac{1}{2} \int_{-1}^{+1} H^{(3)} d\tau, && (18)
\end{aligned}$$

where $\boldsymbol{\lambda}(\tau) \in \mathbb{R}^n$ is the costate, $\gamma(\tau) \in \mathbb{R}$ is the Lagrange multiplier associated with the path constraint of Eq. (8), $\boldsymbol{\upsilon} \in \mathbb{R}^p$ is the Lagrange multiplier associated with the boundary condition of Eq. (4), $\boldsymbol{\omega} \in \mathbb{R}^q$ is the Lagrange multiplier associated with the tangency conditions of Eq. (6), and

$$\begin{aligned}
H^{(k)} &= g^{(k)} + \langle \boldsymbol{\lambda}^{(k)}, \mathbf{f}^{(k)} \rangle, && (k = 1, 3), \\
H^{(k)} &= g^{(k)} + \langle \boldsymbol{\lambda}^{(k)}, \mathbf{f}^{(k)} \rangle - \langle \gamma^{(k)}, \nabla_\tau^q C^{(k)} \rangle, && (k = 2)
\end{aligned} \tag{19}$$

is the augmented Hamiltonian.

III. Radau Pseudospectral Method for Problem $\mathcal{M}$

We now discretize problem $\mathcal{M}$ using the previously developed *Radau pseudospectral method*.[18–20] In the Radau pseudospectral method, the state and its time derivative are approximated, respectively, in each mesh interval $\mathcal{S}_k,\ (k=1,\ldots,K)$, as

$$\begin{aligned} \mathbf{y}^{(k)}\left(\tau^{(k)};T_{k-1},T_k\right) \approx \mathbf{Y}^{(k)}\left(\tau^{(k)}\right) &= \sum_{i=1}^{N_k+1} \mathbf{Y}_i^{(k)} L_i^{(k)}\left(\tau^{(k)}\right), \\ \nabla_{\tau^{(k)}}\mathbf{y}^{(k)}\left(\tau^{(k)};T_{k-1},T_k\right) \approx \nabla_{\tau^{(k)}}\mathbf{Y}^{(k)}\left(\tau^{(k)}\right) &= \sum_{i=1}^{N_k+1} \mathbf{Y}_i^{(k)} \nabla_{\tau^{(k)}} L_i^{(k)}\left(\tau^{(k)}\right), \end{aligned} \tag{20}$$

where $\left(\tau_1^{(k)},\ldots,\tau_{N_k}^{(k)}\right)$ are the LGR points defined on $\tau\in[-1,+1)$ in mesh interval $\mathcal{S}_k$, $\tau_{N_k+1}^{(k)}=+1$ is a non-collocated point, $L_i^{(k)}(\tau),\ (i=1,\ldots,N_k+1)$ is a basis of N_k+1 Lagrange polynomials with support points at $\left(\tau_1^{(k)},\ldots,\tau_{N_k+1}^{(k)}\right)$ and defined as

$$L_i^{(k)}\left(\tau^{(k)}\right) = \prod_{\substack{j=1\\ j\neq i}}^{N_k+1} \frac{\tau^{(k)}-\tau_j^{(k)}}{\tau_i^{(k)}-\tau_j^{(k)}}. \tag{21}$$

Problem $\mathcal{M}$ is then approximated by the following nonlinear programming problem (NLP, defined as problem $\mathcal{N}$). Minimize the cost function

$$\Phi(\mathbf{Y}_1^{(1)},T_0,\mathbf{Y}_{N_3+1}^{(3)},T_3) + \sum_{k=1}^{3}\sum_{j=1}^{N_k} \frac{T_k-T_{k-1}}{2} w_j^{(k)} g(\mathbf{Y}_j^{(k)},\mathbf{U}_j^{(k)},\tau_j^{(k)};T_k,T_{k-1}) \tag{22}$$

subject to the algebraic constraints

$$\begin{aligned} \frac{T_k-T_{k-1}}{2}\mathbf{f}(\mathbf{Y}_i^{(k)},\mathbf{U}_i^{(k)},\tau_i^{(k)};T_k,T_{k-1}) - \sum_{j=1}^{N_k+1} D_{ij}^{(k)}\mathbf{Y}_j^{(k)} &= \mathbf{0}, \quad (i=1,\ldots,N_k), && (23)\\ \nabla_t^q C(\mathbf{Y}_j^{(2)},\mathbf{U}_j^{(2)},\tau_j^{(2)};T_2,T_1) &= 0, \quad (i=1,\ldots,N_2), && (24)\\ \boldsymbol{\phi}(\mathbf{Y}_1^{(1)},T_0,\mathbf{Y}_{N_3+1}^{(3)},T_3) &= \mathbf{0}, && (25)\\ \boldsymbol{\psi}(\mathbf{Y}_1^{(2)},T_1) &= \mathbf{0}, && (26) \end{aligned}$$

where $D_{ij}^{(k)} = \nabla_\tau L_i^{(k)}\left(\tau_j^{(k)}\right),\ (i=1,\ldots,N_k,\ j=1,\ldots,N_k+1)$ is the $N_k\times(N_k+1)$ *Radau pseudospectral differentiation matrix*[18] in mesh interval k, and $w_j^{(k)}\ (j=1,\ldots,N_k)$ are the LGR *weights* in mesh interval k.

A. First-Order Optimality Conditions and Transformed Adjoint System of Problem $\mathcal{N}$

The first-order optimality conditions of problem $\mathcal{N}$ can be written as (see Ref. 36 for details)

$$
\begin{aligned}
\mathbf{0} &= \nabla_U H_j^{(k)}, && (k=1,2,3),\ (j=1,\ldots,N_k), && (27)\\
\mathbf{D}_j^{\dagger(1)}\boldsymbol{\lambda}_{1:N_1}^{(1)} &= -\frac{T_1-T_0}{2}\nabla_Y H_j^{(1)} + \frac{\delta_{1j}}{w_1^{(1)}}\left(-\nabla_{Y_1^{(1)}}\left(\mathbf{\Phi}-\langle\boldsymbol{\upsilon},\boldsymbol{\phi}\rangle\right)-\boldsymbol{\lambda}_1^{(1)}\right), && (j=1,\ldots,N_1), && (28)\\
\mathbf{D}_j^{\dagger(2)}\boldsymbol{\lambda}_{1:N_2}^{(2)} &= -\frac{T_2-T_1}{2}\nabla_Y H_j^{(2)} + \frac{\delta_{1j}}{w_1^{(2)}}\left(\nabla_{Y_1^{(2)}}\langle\boldsymbol{\omega},\boldsymbol{\psi}\rangle+\boldsymbol{\lambda}_{N_1+1}^{(1)}-\boldsymbol{\lambda}_1^{(2)}\right), && (j=1,\ldots,N_2), && (29)\\
\mathbf{D}_j^{\dagger(3)}\boldsymbol{\lambda}_{1:N_3}^{(3)} &= -\frac{T_3-T_2}{2}\nabla_Y H_j^{(3)} - \frac{\delta_{1j}}{w_1^{(3)}}\left(\boldsymbol{\lambda}_{N_2+1}^{(2)}-\boldsymbol{\lambda}_1^{(3)}\right), && (j=1,\ldots,N_3), && (30)\\
\boldsymbol{\lambda}_{N_3+1}^{(3)} &= \nabla_{Y_{N_3+1}^{(3)}}\left(\mathbf{\Phi}-\langle\boldsymbol{\upsilon},\boldsymbol{\phi}\rangle\right), &&&& (31)\\
H_1^{(1)} &= \nabla_{T_0}(\mathbf{\Phi}-\langle\boldsymbol{\upsilon},\boldsymbol{\phi}\rangle), &&&& (32)\\
H_1^{(2)} &= H_{N_1+1}^{(1)} - \nabla_{T_1}\langle\boldsymbol{\omega},\boldsymbol{\psi}\rangle, &&&& (33)\\
H_1^{(3)} &= H_{N_2+1}^{(2)}, &&&& (34)\\
H_{N_3+1}^{(3)} &= -\nabla_{T_3}(\mathbf{\Phi}-\langle\boldsymbol{\upsilon},\boldsymbol{\phi}\rangle), &&&& (35)
\end{aligned}
$$

where

$$
\left.\begin{aligned}
H^{(k)}(T_{k-1}) \approx H_1^{(k)} &= -\frac{T_k-T_{k-1}}{2}\sum_{j=1}^{N_k} w_j^{(k)}\nabla_{T_{k-1}}H_j^{(k)} + \frac{1}{2}\sum_{j=1}^{N_k} w_j^{(k)}H_j^{(k)}\\
H^{(k)}(T_k) \approx H_{N_k+1}^{(k)} &= \frac{T_k-T_{k-1}}{2}\sum_{j=1}^{N_1}\nabla_{T_k} w_j^{(k)}H_j^{(k)} + \frac{1}{2}\sum_{j=1}^{N_k} w_j^{(k)}H_j^{(k)}
\end{aligned}\right\},\quad (k=1,2,3).
$$

Now using the property of $\mathbf{D}^{(k)},\ (k=1,2,3)$ that $\mathbf{D}_{N_k+1}^{(k)} = -\mathbf{D}_{1:N}^{(k)}\mathbf{1}$ (see Ref. 18), where $\mathbf{1}$ is a column vector of all ones, we obtain

$$
\boldsymbol{\lambda}_{N_k+1}^{(k)} = \boldsymbol{\lambda}_1^{(k)} + \frac{T_k-T_{k-1}}{2}\sum_{j=1}^{N_k} w_j^{(k)}\mathbf{D}_j^{\dagger(k)}\boldsymbol{\lambda}^{(k)}. \tag{36}
$$

Combining Eqs. (28)–(30) with Eq. (36), we obtain

$$
\begin{aligned}
\boldsymbol{\lambda}_{N_1+1}^{(1)} &= -\nabla_{Y_1^{(1)}}\left(\mathbf{\Phi}+\langle\boldsymbol{\upsilon},\boldsymbol{\phi}\rangle\right) - \frac{T_1-T_0}{2}\sum_{j=1}^{N_1} w_j^{(1)}\nabla_Y H^{(1)}(\mathbf{Y}_j,\mathbf{U}_j,\boldsymbol{\lambda}_j,\gamma_j), && (37)\\
\boldsymbol{\lambda}_{N_2+1}^{(2)} &= \nabla_{Y_1^{(2)}}\langle\boldsymbol{\omega},\boldsymbol{\psi}\rangle + \boldsymbol{\lambda}_{N_1+1}^{(1)} - \frac{T_2-T_1}{2}\sum_{j=1}^{N_2} w_j^{(2)}\nabla_Y H^{(2)}(\mathbf{Y}_j,\mathbf{U}_j,\boldsymbol{\lambda}_j,\gamma_j), && (38)\\
\boldsymbol{\lambda}_{N_3+1}^{(3)} &= \boldsymbol{\lambda}_{N_2+1}^{(2)} - \frac{T_3-T_2}{2}\sum_{j=1}^{N_3} w_j^{(3)}\nabla_Y H^{(3)}(\mathbf{Y}_j,\mathbf{U}_j,\boldsymbol{\lambda}_j,\gamma_j), && (39)
\end{aligned}
$$

In Eqs. (37)–(39) a Legendre-Gauss-Radau quadrature is used as an approximation of the integral of the costate dynamics, that is,

$$
\boldsymbol{\lambda}^{(k)}(T_k) = \boldsymbol{\lambda}^{(k)}(T_{k-1}) + \frac{T_k-T_{k-1}}{2}\int_{-1}^{+1} -\nabla_y H^{(k)}d\tau \Longleftrightarrow \boldsymbol{\lambda}_{N_k+1}^{(k)} \approx \boldsymbol{\lambda}_1^{(k)} - \frac{T_k-T_{k-1}}{2}\sum_{j=1}^{N_k} w_j^{(k)}\nabla_Y H_j^{(k)}
$$

has been used in each mesh interval. Thus, from Eqs. (37)–(39) we obtain

$$\boldsymbol{\lambda}^{(1)}(T_0) \approx \boldsymbol{\lambda}_1^{(1)} = -\nabla_{Y_1^{(1)}}\left(\boldsymbol{\Phi} - \langle \boldsymbol{\upsilon}, \boldsymbol{\phi}\rangle\right), \tag{40}$$

$$\boldsymbol{\lambda}^{(2)}(T_1) \approx \boldsymbol{\lambda}_1^{(2)} = \boldsymbol{\lambda}^{(1)}(T_1) + \nabla_{Y_1^{(2)}}\langle \boldsymbol{\omega}, \boldsymbol{\psi}\rangle, \tag{41}$$

$$\boldsymbol{\lambda}^{(3)}(T_2) \approx \boldsymbol{\lambda}_1^{(3)} = \boldsymbol{\lambda}^{(2)}(T_2). \tag{42}$$

Equations (40)–(42) show that the second terms on the right-hand sides of Eqs. (28)–(30) are approximately zero. This shows an equivalence between the transformed adjoint system of the finite dimensional NLP given by problem $\mathcal{N}$ and the first-order optimality conditions of the continuous-time optimal control problem $\mathcal{M}$.

IV. Example

Consider the following optimal control problem, denoted $\mathcal{E}$, obtained from Ref. 26:

$$\mathcal{E}: \left\{ \text{Minimize } \tfrac{1}{2}\int_0^1 u^2 dt \quad \text{subject to} \quad \left[\begin{array}{lcl} \nabla_t x & = & v, \\ \nabla_t v & = & u, \\ x(0) & = & x(1) = 0, \\ v(0) & = & -v(1) = 1, \\ x(t) & \leq & \ell. \end{array}\right.\right. \tag{43}$$

It is seen that the second derivative of the state-inequality path constraint $x(t) \leq \ell$ is an explicit function of the control variable. Consequently, the original state-inequality path constraint can be replaced by the tangency conditions

$$\boldsymbol{\psi}(x(T_1), T_1) = \begin{bmatrix} x(T_1) - \ell \\ v(T_1) \end{bmatrix} = \mathbf{0} \tag{44}$$

and control equality path constraint

$$u(t) = 0, \qquad t \in [T_1, T_2]. \tag{45}$$

The modified version of problem $\mathcal{E}$, denoted problem $\mathcal{F}$ is then given as follows:

$$\mathcal{F}: \left\{ \text{Minimize } \tfrac{1}{2}\int_0^1 u^2 dt \quad \text{subject to} \quad \left[\begin{array}{lcl} \nabla_t x & = & v, \\ \nabla_t v & = & u, \\ x(0) & = & x(1) = 0, \\ v(0) & = & -v(1) = 1, \\ x(T_1) & = & 0, \\ v(T_1) & = & 0, \\ u(t) & = & 0 \qquad t \in [T_1, T_2]. \end{array}\right.\right. \tag{46}$$

It is known for this example that the inequality path constraint is inactive for $\ell > 1/4$, is active at only a single point for $1/6 < \ell \leq 1/4$, and is active along a nonzero duration arc for $0 < \ell \leq 1/6$.

In the case where $0 < \ell \le 1/6$, the optimal solution is

$$x^*(t) = \begin{cases} \ell\left[1 - \left(1 - \frac{t}{3\ell}\right)^3\right], & 0 \le t \le 3\ell, \\ \ell, & 3\ell \le t \le 1 - 3\ell, \\ \ell\left[1 - \left(1 - \frac{1-t}{3\ell}\right)^3\right], & 1 - 3\ell \le t \le 1, \end{cases} \qquad \lambda_x^*(t) = \begin{cases} \frac{2}{9\ell^2}, & 0 \le t \le 3\ell, \\ -\frac{2}{9\ell^2}, & 3\ell \le t \le 1, \end{cases}$$

$$v^*(t) = \begin{cases} \left(1 - \frac{t}{3\ell}\right)^2, & 0 \le t \le 3\ell, \\ 0, & 3\ell \le t \le 1 - 3\ell, \\ -\left(1 - \frac{1-t}{3\ell}\right)^2, & 1 - 3\ell \le t \le 1, \end{cases} \qquad \lambda_v^*(t) = \begin{cases} \frac{2}{3\ell}\left(1 - \frac{t}{3\ell}\right), & 0 \le t \le 3\ell, \\ \frac{2}{3\ell}\left(1 - \frac{1-t}{3\ell}\right), & 3\ell \le t \le 1, \end{cases},$$

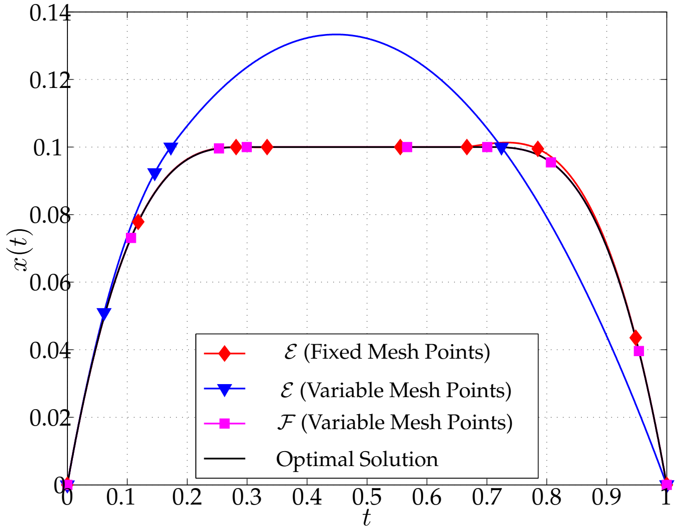
$$u^*(t) = \begin{cases} -\frac{2}{3\ell}\left(1 - \frac{t}{3\ell}\right), & 0 \le t \le 3\ell, \\ 0, & 3\ell \le 1 - 3\ell, \\ -\frac{2}{3\ell}\left(1 - \frac{1-t}{3\ell}\right), & 1 - 3\ell \le t \le 1 \end{cases}$$

A. Solutions to Problems $\mathcal{E}$ and $\mathcal{F}$

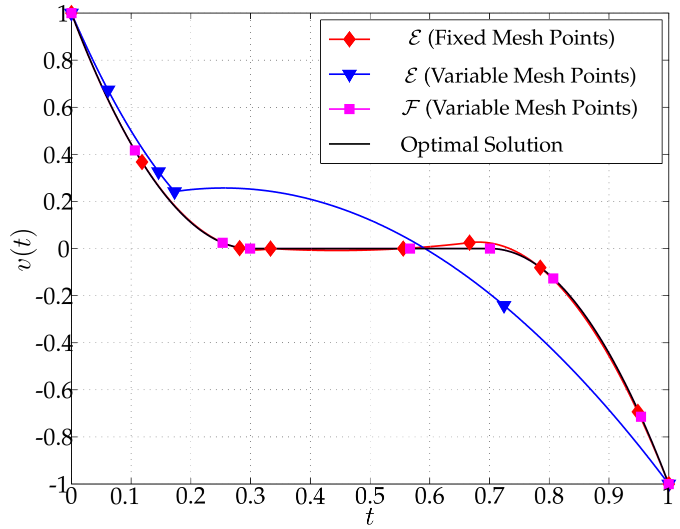
Solutions are now presented to problems $\mathcal{E}$ and $\mathcal{F}$ given in Eqs. (43) and (46), respectively), using the version of the Radau pseudospectral method presented in Section III. In order to compare the results obtained using the method developed in this paper against other standard formulations, problem $\mathcal{E}$ was solved using both fixed and variable mesh points (referred to henceforth as problems "$\mathcal{E}_F$" and "$\mathcal{E}_V$") while problem $\mathcal{F}$ [that is, the formulation developed in this paper and referred to henceforth as problem "$\mathcal{F}_V$"] was solved using variable mesh points. When fixed mesh points were used, the interior mesh points were placed at $T_1 = 1/3$ and $T_2 = 2/3$. When variable mesh points were used, T_1 and T_2 were variables in both the continuous optimal control problem and the Radau pseudospectral discrete approximation. Because the optimal solution to this problem is a piecewise cubic, quadratic, and cubic polynomial in the first, second, and third mesh intervals, respectively, all three problems were solved using $N_1 = 3$, $N_2 = 2$, and $N_3 = 3$ LGR points. All problems were solved using the NLP solver SNOPT.[37] In all results that follows the state and costate are approximated using the piecewise Lagrange interpolating polynomial approximations.

Figure 1 shows the state and control for all three aforementioned problems. From these results, it is seen that even though the discrete solution to problem $\mathcal{E}_V$ satisfies all the constraints at the *collocation points*, it is highly inaccurate and does not satisfy the inequality path constraint $x(t) \le \ell$ once it is interpolated. Next, it is seen that the solution of problem $\mathcal{F}_V$ which corresponds to the method developed in this paper presented in Section III, produces an accurate state, control, and costate. Importantly, it is seen that including the tangency conditions and treating the mesh points as variables in the nonlinear programming problem (NLP), an accurate approximation to the start and terminus times of the path constraint activity is obtained, therefore capturing the discontinuities in the solution.

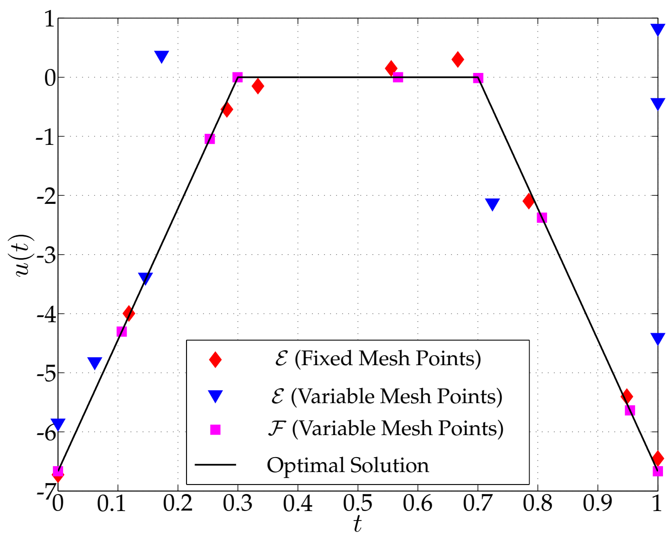
It can be shown that the discontinuity in the optimal costate is not unique. Specifically, this discontinuity can occur at either the entrance, exit, or both the entrance and the exit of a constrained arc. The problem formulation described in this paper (namely, problem $\mathcal{F}_V$ in the case of this example), uniquely defines the costate discontinuity in such a manner that it will always occurs at only the start of the constrained arc. Thus, the method developed in this paper leads to solutions that resemble those described in Ref. 26. When solving problem $\mathcal{E}$, however, the discontinuity in the optimal costate is *not* uniquely defined and, as a result, it may be possible that the NLP solver will converge to one of the other possible solutions. Figure 2 shows the costate obtained for both these possible solutions. In particular, it is seen that the optimal costate obtained when solving problem $\mathcal{F}_V$ contains a single discontinuity at the entrance of the constrained arc, while the optimal costate obtained when solving problem $\mathcal{E}$ contains discontinuities at both the entrance and exit of the constrained arc.



(a) $x(t)$ vs. t.

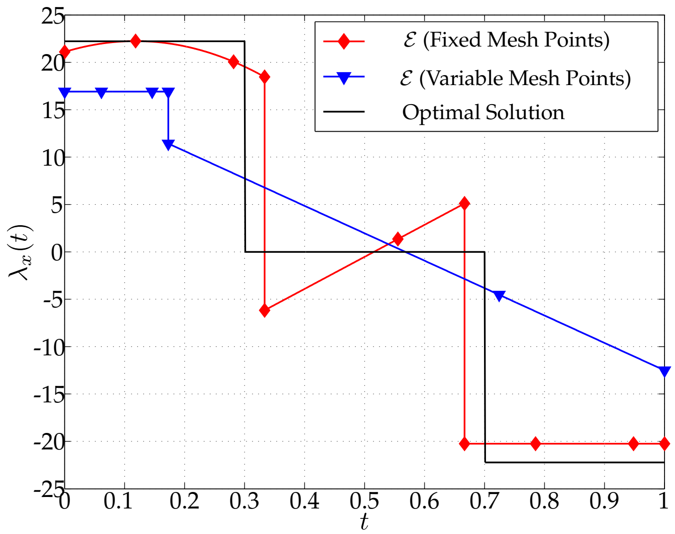


(b) $v(t)$ vs. t

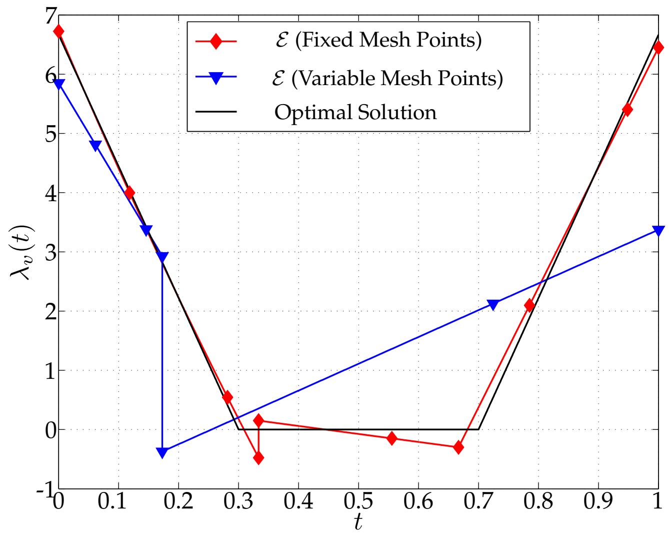


(c) $u(t)$ vs. t

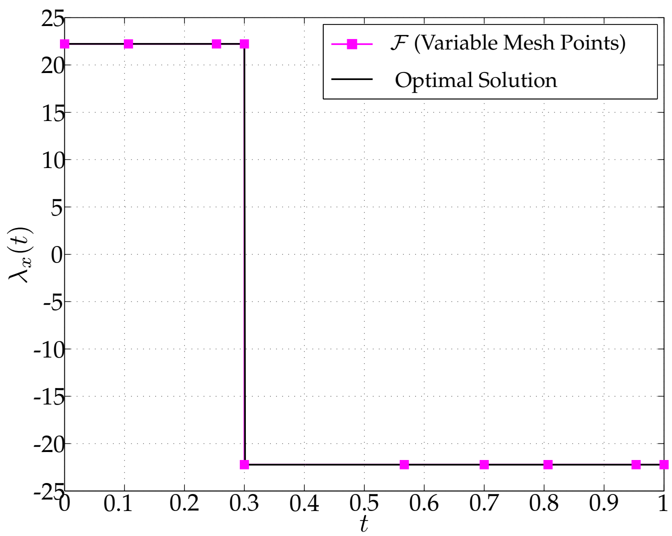
Figure 1: Radau pseudospectral solutions of the state, $(x(t), v(t))$, and control $u(t)$, for example obtained by solving problems $\mathcal{E}_F$, $\mathcal{E}_V$, and $\mathcal{F}_V$.



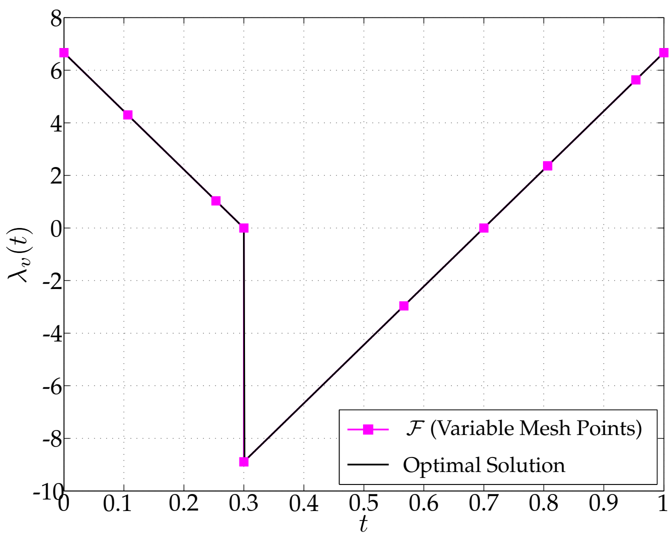
(a) $\lambda_x(t)$ vs. t for problem $\mathcal{E}_F$ and $\mathcal{E}_V$.



(b) $\lambda_v(t)$ vs. t for problem $\mathcal{E}_F$ and $\mathcal{E}_V$.



(c) $\lambda_x(t)$ vs. t for problem $\mathcal{F}$.



(d) $\lambda_v(t)$ vs. t for problem $\mathcal{F}$.

Figure 2: Radau pseudospectral solutions of the costate, $(\lambda_x(t), \lambda_v(t))$, for example obtained by solving problems $\mathcal{E}_F$, $\mathcal{E}_V$, and $\mathcal{F}_V$.

V. Conclusions

A direct collocation Radau pseudospectral method has been developed to discretize a state-inequality path constrained continuous-time optimal control problem. It was shown that by modifying the original state-inequality path constrained, adding a set of conditions that define the start of the constrained arc, and treating the mesh points as variables, the first-order optimality conditions of the Radau pseudospectral nonlinear programming problem are a discrete form of the first-order conditions obtained from the calculus of variations. A classic state-inequality constrained optimal control problem was studied in detail to demonstrate the improvement in the accuracy obtained using the approach developed in this paper over an unmodified Radau pseudospectral method.

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